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CONCAVE SHELLS OF CONTINUITY MODULES

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We prove the inequality

$$\overline{\omega}(t) \le \inf_{s>0}\left(\omega\left(\frac{s}{2}\right) + \frac{\omega(s)}{s}t\right),$$

where $\omega(t)$ is a function of the modulus-of-continuity type and $\overline{\omega}(t)$ is its smallest concave majorant. The consequences obtained for Jackson's inequalities in $C_{2\pi}$ are presented.

Let $\omega(t)\colon R^+ \to R^+$ be a function of the modulus-of-continuity type, i.e., $\omega(t)$ is a continuous nondecreasing function, $\omega(0) = 0$, and $\omega(t_1 + t_2) \le \omega(t_1) + \omega(t_2)$. Also let Ω be the class of all functions of this kind. The following lemma is true for the least concave majorant $\overline{\omega}(t)$:

Lemma. *For any $\omega \in \Omega$ and all $k \in N$, the inequalities*

$$\overline{\omega}(kt) \le (k+1)\omega(t) \tag{1}$$

are true. Inequality (1) is exact on the class Ω, i.e., for any $t > 0$,

$$\sup_{\omega\in\Omega}\frac{\overline{\omega}(kt)}{\omega(t)} = k+1. \tag{2}$$

Earlier, this lemma was proved by Stechkin [1] for $k = 1$ and by Korneichuk [2] for $k \in N$. Let

$$\omega(f,h) := \max_{|t|\le h}\max_x |f(x+t) - f(x)| = \max_{|t|\le h}\|f(\cdot+t) - f(\cdot)\|$$

be the modulus of continuity of a 2π-periodic continuous function f in the space $C_{2\pi}$ and let

$$\|f\| = \max_x |f(x)|.$$

Then $\omega(f,h) \in \Omega$ and, in addition, the property

$$\omega(f,h) = \omega(f,\pi) \tag{3}$$

is true for all $h \ge \pi$.

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Assume that the class Ω contains only functions ω for which the additional property (3) is true. For any ω of this type from $\Omega,$ there exists a function $f \in C_{2\pi}$ such that [3] (Sec. 7.1)

$$\omega(f,t) = \omega(t) \tag{4}$$

for all $t > 0.$

We prove a somewhat corrected inequality (1).

Theorem. *Suppose that $\omega \in \Omega$. Then, for all $t > 0$,*

$$\overline{\omega}(t) \le \inf_{s>0} \left(\omega\left(\frac{s}{2}\right) + \frac{\omega(s)}{s} t \right) \tag{5}$$

and, in particular,

$$\overline{\omega}(kt) \le \omega\left(\frac{t}{2}\right) + k\omega(t). \tag{6}$$

For all $k \in N$ and every $t \in \left(0, \dfrac{\pi}{k}\right)$, inequality (6) is unimprovable on the class Ω in a sense that

$$\sup_{\omega\in\Omega} \frac{\overline{\omega}(kt)}{\omega\left(\dfrac{t}{2}\right) + k\omega(t)} = 1. \tag{7}$$

Proof. By the Peetre theorem [4],

$$\frac{1}{2}\overline{\omega}(f,2t) = K(f,t;C,C^1) := \inf_{g\in C^1}(\|f-g\| + t\|g'\|) = \inf_{N>0}\{\|f-g\| + tN; \|g'\| \le N\}. \tag{8}$$

According to the Korneichuk theorem [3] (Sec. 8.3), we get

$$\inf\{\|f-g\|; \|g'\| \le N\} = \frac{1}{2}\max_{y\in[0,\pi]}(\omega(f,y) - Ny). \tag{9}$$

If follows from (4), (8), and (9) that

$$\overline{\omega}(t) = \inf_{N>0}\left(\max_{y\in[0,\pi]}(\omega(y) - Ny) + Nt\right).$$

For any $s \in (0,\pi),$ we set

$$N = \frac{\omega(s)}{s}.$$

Then

$$\overline{\omega}(t) \le \inf_{s}\left(\max_{y\in[0,\pi]}\left(\omega(y) - \frac{\omega(s)}{s}y\right) + \frac{\omega(s)}{s}t\right). \tag{10}$$

Note that

$$\max_{y\in[0,\pi]}\left(\omega(y)-\frac{\omega(s)}{s}y\right)=\max_{y\in[0,s]}\left(\omega(y)-\frac{\omega(s)}{s}y\right). \tag{11}$$

Indeed, let $y>s$, i.e., $y=ks+y'$, where $k\in N$ and $y'\in[0,s]$. Then

$$\begin{aligned}\omega(y)-\frac{\omega(s)}{s}y&=\omega(ks+y')-\frac{\omega(s)}{s}(ks+y')\\&\le(k\omega(s)+\omega(y'))-\left(k\omega(s)+\frac{\omega(s)}{s}y'\right)\\&=\omega(y')-\frac{\omega(s)}{s}y'.\end{aligned}$$

We now show that

$$\max_{y\in[0,s]}\left(\omega(y)-\frac{\omega(s)}{s}y\right)\le\omega\Big(\frac{s}{2}\Big). \tag{12}$$

For $y\in\Big[0,\dfrac{s}{2}\Big]$, this is obvious. Let $y\in\Big[\dfrac{s}{2},s\Big]$. Then

$$\omega(y)-\frac{\omega(s)}{s}y\le\omega(s)-\frac{\omega(s)}{s}\frac{s}{2}=\frac{1}{2}\omega\left(2\cdot\frac{s}{2}\right)\le\omega\Big(\frac{s}{2}\Big).$$

In view of the arbitrariness of s, inequality (5) follows from (10)–(12).

Since

$$\omega\left(\frac{t}{2}\right)+k\omega(t)\le(k+1)\omega(t),$$

relation (7) follows from (2):

$$\sup_{\omega\in\Omega}\frac{\overline{\omega}(kt)}{\omega\left(\dfrac{t}{2}\right)+k\omega(t)}\ge\sup_{\omega\in\Omega}\frac{\overline{\omega}(kt)}{(k+1)\omega(t)}=1.$$

The theorem is proved.

Relation (2) appears to be useful in proving the exact Jackson inequalities for the best uniform approximations of continuous periodic functions by trigonometric polynomials. If

$$e_{n-1}(f):=\inf_{\{C_k\}}\left\|f(x)-\sum_{|k|\le n-1}C_ke^{ikx}\right\|,$$

then, by the Korneichuk theorem [3] (Sec. 7.6), we get

$$e_{n-1}(f)\le\frac{1}{2}\overline{\omega}\Big(f,\frac{\pi}{n}\Big). \tag{13}$$

It follows from (2) that, for $k \in N,$ we can write

$$e_{n-1}(f) \le \frac{k+1}{2}\omega\Big(f, \frac{\pi}{nk}\Big).$$

For any $k \in N,$ this inequality is uniformly exact in $n,$ namely [2],

$$\left(1 - \frac{1}{2n}\right)\frac{1}{2} \le \sup_{f \in C_{2\pi}} \frac{e_{n-1}(f)}{(k+1)\omega\Big(f, \dfrac{\pi}{nk}\Big)} \le \frac{1}{2}. \tag{14}$$

If, instead of (2), we apply relation (5) to inequality (13), then we get the following form of the Jackson inequality:

$$e_{n-1}(f) \le \frac{1}{2}\inf_{s>0}\left(\omega\Big(f, \frac{s}{2}\Big) + \frac{\omega(f,s)}{s}\frac{\pi}{n}\right). \tag{15}$$

We now mention some specific values of s for which the constant $\dfrac{1}{2}$ on the right-hand side of (15) is unimprovable:

$$\left(1 - \frac{1}{2n}\right)\frac{1}{2} \le \sup_{f \in C_{2\pi}} \frac{e_{n-1}(f)}{\omega\Big(f, \dfrac{\pi}{n}\Big) + \dfrac{1}{2}\omega\Big(f, \dfrac{2\pi}{n}\Big)} \le \frac{1}{2},$$

for $k \in N,$

$$\left(1 - \frac{1}{2n}\right)\frac{1}{2} \le \sup_{f \in C_{2\pi}} \frac{e_{n-1}(f)}{\omega\Big(f, \dfrac{\pi}{2nk}\Big) + k\omega\Big(f, \dfrac{\pi}{nk}\Big)} \le \frac{1}{2}.$$

In particular,

$$\left(1 - \frac{1}{2n}\right)\frac{1}{2} \le \sup_{f \in C_{2\pi}} \frac{e_{n-1}(f)}{\omega\Big(f, \dfrac{\pi}{2n}\Big) + \omega\Big(f, \dfrac{\pi}{n}\Big)} \le \frac{1}{2}. \tag{16}$$

Here, the lower bounds directly follow from (14).

Note that relations similar to (16) with the same constant $1/2$ are also true in the spaces $L_p[0, 2\pi],\ p \in [1, 2].$

Let

$$\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right)^{1/p},$$

$$e_{n-1}(f)_p := \inf_{\{C_k\}} \left\| f(x) - \sum_{|k| \le n-1} C_k e^{ikx} \right\|_p,$$

$$\omega(f,h)_p = \sup_{|t| \le h} \|\Delta_t f(x)\|_p, \qquad \Delta_t f(x) = f(x+t) - f(x).$$

In [5, 6], Chernykh proved the following Jackson inequalities sharp for all $n \in N$:

$$\begin{gathered} e_{n-1}(f)_2 \le \frac{1}{2^{1/2}} \omega\Big(f, \frac{\pi}{n}\Big)_2, \\ e_{n-1}(f)_p \le \frac{1}{2^{1-\frac{1}{p}}} \omega\left(f, \frac{2\pi}{n}\right)_p, \quad p \in [1, 2). \end{gathered} \tag{17}$$

These inequalities follow from his more exact inequalities:

$$\begin{gathered} e_{n-1}^2(f)_2 \le \frac{n}{4} \int\limits_0^{\pi/n} \sin nt \|\Delta_t f\|_2^2 dt, \\ e_{n-1}^p(f)_p \le \frac{1}{2^{p-1}} \frac{n}{4} \int\limits_0^{2\pi/n} \sin \frac{n}{2} t \|\Delta_t f\|_p^p \, dt, \quad p \in [1, 2). \end{gathered} \tag{18}$$

Since

$$\frac{n}{4} \int\limits_0^{\pi/n} \sin nt \|\Delta_t f\|_2^2 \, dt = \frac{n}{4} \int\limits_0^{\pi/2n} \sin nt \|\Delta_t f\|_2^2 \, dt + \frac{n}{4} \int\limits_{\pi/2n}^{\pi/n} \sin nt \|\Delta_t f\|_2^2 \, dt$$

$$\le \frac{1}{4} \omega^2\Big(f, \frac{\pi}{2n}\Big)_2 + \frac{1}{4} \omega^2\Big(f, \frac{\pi}{n}\Big)_2,$$

we have

$$e_{n-1}(f)_2 \le \frac{1}{2} \Big(\omega^2 \Big(f, \frac{\pi}{2n}\Big)_2 + \omega^2 \Big(f, \frac{\pi}{n}\Big)_2\Big)^{1/2}. \tag{19}$$

Similarly, for $p \in [1, 2)$, we get

$$e_{n-1}(f)_p \le \frac{1}{2} \left(\omega^p \Big(f, \frac{\pi}{n}\Big)_p + \omega^p \left(f, \frac{2\pi}{n}\right)_p\right)^{1/p}. \tag{20}$$

The constant $1/2$ in inequalities (19) and (20) is sharp in $L_p[0, 2\pi]$ for any n, and the extreme functions are the same as in (17) (see [5, 6]).

For $p \in (2, \infty)$, the exact inequalities similar to (17) and (18) are known only for $n = 1$. Thus, the inequality

$$e_0(f)_p \le \frac{1}{2^{1/p}} \omega(f, \pi)_p$$

was obtained in [7] and the inequality

$$e_0(f)_p \le \frac{1}{2^{1/p}} \left(\frac{1}{\pi} \int\limits_0^{\pi} \|\Delta_t f\|_p^{p'} dt\right)^{1/p'}, \tag{21}$$

where $p' = p(p-1)^{-1}$, was deduced in [8]. Inequality (21) yields the following analog of the exact inequalities (19) and (20) for $n = 1$ and $p > 2$:

$$e_0(f)_p \le \frac{1}{2^{1/p}} \left(\frac{1}{\pi} \int_0^{\pi/2} \omega^{p'}(f,t)_p dt + \frac{1}{\pi} \int_{\pi/2}^{\pi} \omega^{p'}(f,t)_p dt \right)^{1/p'}$$

$$\le \frac{1}{2} \left(\omega^{p'}\Big(f, \frac{\pi}{2}\Big)_p + \omega^{p'}(f,\pi)_p \right)^{1/p'}. \tag{22}$$

A sequence of δ-shaped functions is extreme in (22).

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