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ON THE JACKSON THEOREM FOR PERIODIC FUNCTIONS IN METRIC SPACES WITH INTEGRAL METRIC. II

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In the spaces $L_\psi(T^m)$ of periodic functions with metric

$$\rho(f,0)_\psi = \int_{T^m} \psi(|f(x)|)\,dx,$$

where ψ is a function of the type of modulus of continuity, we study the direct Jackson theorem in the case of approximation by trigonometric polynomials. It is proved that the direct Jackson theorem is true if and only if the lower dilation index of the function ψ is not equal to zero.

1. Introduction

The present paper is a continuation of [1].

Let Ω be the class of functions $\psi: R_+^1 \to R_+^1$ that are moduli of continuity, i.e., ψ is a continuous nondecreasing function, $\psi(0) = 0$, and $\psi(x+y) \le \psi(x) + \psi(y)$ for all $x, y \in R_+^1$. Assume that $f(x)$, $x \in R^1$, are real-valued functions of period 1, $\mathbf{T} = [-1/2, 1/2)$ is the basic torus of periods, $L_0 \equiv L_0(\mathbf{T})$ is the set of all functions of this type that are finite and measurable almost everywhere on $\mathbf{T}$, and, for $\psi \in \Omega$, the set

$$L_\psi \equiv L_\psi(T) = \left\{ f \in L_0(T) : \|f\|_\psi := \int_T \psi(|f(x)|)\,dx < \infty \right\}$$

is a linear metric space with metric $\rho(f,g)_\psi = \|f-g\|_\psi$.

In particular, using the function $\phi(t) = t(1+t)^{-1}$, $\phi \in \Omega$, we introduce the metric

$$\rho(f,g)_0 := \int_T \phi(|f(x) - g(x)|)\,dx$$

in L_0, which generates convergence in measure. In the case where $\psi(t) = t^p$, $0 < p < 1$, we obtain the spaces L_p.

Let

$$T_n(x) = \sum_{k=-n}^{n} c_k e^{i2\pi kx}$$

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be a real-valued trigonometric polynomial of period 1 and degree n, let

$$E_n(f)_\psi \ := \ \inf_{\{c_k\}} \left\| f - T_n \right\|_\psi$$

be the best approximation of f by these polynomials in the space L_ψ, and let

$$\omega(f,h)_\psi \ := \ \sup\left\{ \left\| \Delta_t f \right\|_\psi : |t| \le h \right\}, \quad h \in R_+^1,$$

be the modulus of continuity of f from L_ψ; here, $\Delta_t f(x) = f(x+t) - f(x)$.

In the theory of approximation of periodic functions, the relations

$$\sup_{n>0} \sup_{\substack{f \in L_\psi \\ f \ne \text{const}}} \frac{E_{n-1}(f)_\psi}{\omega\left(f, \frac{1}{n}\right)_\psi} \ < \ \infty, \tag{1}$$

if they are true, are usually called the Jackson inequalities (or the Jackson theorem).

For information and bibliography on inequalities (1) in the spaces L_ψ, see [1]. Since the Jackson inequalities (1) hold in the spaces L_p and do not hold in the space L_0, the problem of description of functions ψ from Ω for which relations (1) hold in the corresponding spaces L_ψ was posed in [1].

In this direction, the following particular result was proved in [1] (Theorem 2): If a function $\psi \in \Omega$ satisfies the conditions

(i) $\exists M \quad \forall x, y \in R_+^1: \ \psi(x \cdot y) \le M \psi(x)\psi(y)$,

(ii) $\exists \varepsilon > 0: \ \int_0^1 \frac{\psi(t)}{t^{1+\varepsilon}}\, dt < \infty$,

then the Jackson theorem (1) is true in L_ψ.

Note that the first results related to this problem were obtained in [2]. Namely, it was proved in [2] (Theorem 4.3) that if $\psi \in \Omega$ is such that, for a certain $r = 1, 2, \ldots$, one has

$$\sum_{k=1}^{\infty} k M_\psi(k^{-2r}) \ < \ \infty,$$

where

$$M_\psi(c) \ = \ \sup_{x>0} \frac{\psi(cx)}{\psi(x)}, \quad c > 0,$$

then inequalities (1) are true.