

STUDY OF THE DYNAMICS OF PRODUCT SALES PROCESS WITH THE HELP OF ZOLOTAS MODEL

Vasiliy Ye. Belozyorov*, Svetlana A. Volkova†

Abstract. A new mathematical model describing the dynamics of sales of goods on the market has been proposed. This model takes into account the following characteristics: the average welfare of buyers, the maximum welfare of buyers, the number of buyers who know about the incoming product and the level of market saturation with this product. Examples demonstrating the features of the constructed model are given.

Key words: process of product sales, Zolotas model, system of ordinary autonomous differential equations, stability, limit cycle.

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1. Introduction

Due to the widespread dissemination of information technologies in mathematical modeling, new directions have emerged that make it possible to more accurately study economic processes [1, 4, 5, 7, 8, 11]. At the same time, models that have long been known in some areas of economics have received new applications in other areas. The transformation of one such well-known model into another economic model is precisely the subject of research in this article.

Normal social security of society in any country allows us to predict the future of this country and take actions leading to an increase in well-being. The famous Greek economist X. Zolotas [10] proceeds from the hypothesis that increasing the production of a large number of goods does not necessarily lead to an improvement in the quality of life. In order to show how social welfare changes, he considers two types of factors that act with greater or lesser relative intensity depending on the already achieved social state. Some factors stimulate, while others restrain the growth of welfare. It is on the opposition of such factors that the Zolotas model is built.

Let us briefly recall the mathematical formulation of the Zolotas model.

We denote the level of social welfare of society through V , and through A the critical value of this welfare. Then the restraining factor will be $(A - V)$, and the

*Department of Applied Mathematics, Oles Honchar Dnipro National University, Nauka's Avenue, 72, 49045, Dnipro, Ukraine, belozvye2017@gmail.com

†Department of Computer Information Technologies, Ukrainian State University of Science and Technologies, Academician Lazaryan's Street, 2, 49010, Dnipro, Ukraine, svolkovav2017@gmail.com

stimulating factor will be λV . With this approach, the dynamics of social welfare (the Zolotas model) is determined by the equation

$$\frac{dV(x)}{dx} = \lambda V(x) \cdot (A - V(x)), \quad (1.1)$$

where $x > 0$ is the per capita approach; $\lambda > 0$ and $A > 0$.

The solution of equation (1.1) can be represented as

$$V(x) = \frac{A}{1 + (A/V(x_0) - 1) \cdot \exp(-\lambda Ax)}, \quad (1.2)$$

where $A \geq V(x_0) > 0$ is the initial value [9].

Investigating the solution (1.2) of equation (1.1), Zolotas identifies three stages of development of society: I. Society of need; II. Society for Continuous Improvement; III. Society of Crime.

At the present stage of economic development of industrial countries, the restraining factor (I or III) is comparatively stronger than factor II. As a result, the time required for society to rise from the already achieved very high level of well-being to the maximum possible (factor II) is gradually increasing. Reaching the critical value of welfare Zolotas connects both the form of distribution of income and wealth, the degree and speed of natural resources, and the degree of environmental pollution. If we trace the level of social welfare $V(x(t))$ and the function $x(t)$ (in (1.2) $x = x(t)$ is a function of time), then from solution (1.2) we can determine the indicators λA and $V(x_0)$. Then it will be possible to consider the stages of development of the welfare of a particular state and determine the stage in which it is currently located, as well as the distance separating it from the critical point A . (The point $A = A^*$ is called critical if there exists a moment t^* such that $A^* = V(x(t^*))$). It is clear that the point A^* can be reached only if $V(x(t)) \rightarrow A$ at $t \rightarrow \infty$.)

The Zolotas model is the basis of the next economic model, the construction of which we are starting.

Let us consider the process of selling some goods about which at time t , out of the number of potential buyers N , only $P_3 < N$ buyers know about these goods. To speed up the sale of products, an advertising campaign was launched. Let us take into account that after advertising, the rate of change in the number of people who know about the product is proportional to both the number of potential buyers P_3 who already know about the product and the number of buyers $(N - P_3)$ who do not yet know about it. A mathematical model for describing the number of buyers $P_3(t)$ who know about the product, provided that time is counted from the moment the advertisement is released, can be written as follows:

$$\frac{dP_3(t)}{dt} = KN P_3(t) - KMP_3^2(t), \quad (1.3)$$

where $K, M \in \mathbb{R}$ is the coefficient of proportionality. (Note that if $\lambda = K$ and $M = 1$, then equation (1.3) in its structure completely coincides with equation (1.1).)

Let us introduce the following notation: $\mu_0 = -KN$ and $M = -\epsilon N$. Then equation (1.3) can be written as

$$\frac{dP_3(t)}{dt} = -\mu_0 P_3(t) - \epsilon \mu_0 P_3^2(t). \quad (1.4)$$

The purchasing power of a particular individual changes depending on the level of critical welfare of people P_2 according to the law: λP_2 , $\lambda > 0$. Then, taking into account this law, equation (1.4) can be rewritten as follows

$$\frac{dP_3(t)}{dt} = \lambda P_2(t) - \mu_0 P_3(t) - \epsilon \mu_0 P_3^2(t). \quad (1.5)$$

Let P_1 be the average level of social welfare, and P_2 its critical level. Then the value $P_2 - P_1$ is a restraining factor, and $-\epsilon \mu_0 P_3$ is a stimulating factor. In this case the dynamics of the level of social welfare is determined by a differential equation of the form:

$$\frac{dP_1(t)}{dt} = -(\lambda + \mu_0)P_1(t) + \mu_0 P_2(t) - \epsilon \mu_0 P_3(t)(P_1(t) - P_2(t)). \quad (1.6)$$

Now, if in equation (1.6) we also take into account the market saturation factor λP_0 , then we arrive at the equation

$$\frac{dP_1(t)}{dt} = \lambda P_0(t) - (\lambda + \mu_0)P_1(t) + \mu_0 P_2(t) - \epsilon \mu_0 P_3(t)(P_1(t) - P_2(t)). \quad (1.7)$$

In turn, the amount of market saturation with this product P_0 , taking into account the well-being of people P_1 and the number of people P_3 who know about the product, looks like this

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu_0 P_1(t) + \epsilon \mu_0 P_3(t)P_1(t). \quad (1.8)$$

Finally, the equation for the critical welfare of people P_2 , taking into account the number of goods on the market, the general average welfare of people P_1 and taking into account advertising P_3 , can be written as follows:

$$\frac{dP_2(t)}{dt} = \lambda P_1(t) - (\lambda + \mu_0)P_2(t) + \mu_0 P_3(t) - \epsilon \mu_0 P_3(t)(P_2(t) - P_3(t)). \quad (1.9)$$

Thus, we have obtained a closed system (1.8), (1.7), (1.9), and (1.5) of four differential equations that describe the dynamics of sales of a certain product on the market. A mathematical study of this system will be carried out in the following sections.

Comment. The resulting equations contain four types of variables: P_0 and P_3 are functions describing the numbers of certain groups of buyers; P_1 and P_2 are functions describing the welfare of buyers. Naturally, the parameters included in the equations (1.8), (1.7), (1.9), and (1.5) must be selected so that all variables

P_0, \dots, P_3 would be of the same measurement. In what follows, we will assume that the variables P_0, \dots, P_3 describe the numbers of different groups of buyers. In this case, the variable P_1 means the number of people with average welfare, and the variable P_2 means the number of people with critical welfare. (From here it follows that the parameter $\lambda = \psi(N)$ is a function of N .)

2. Mathematical preliminaries

We formalize the variables that are included in the system of equations (1.8), (1.7), (1.9), and (1.5):

P_0 is the quantity of goods presented on the market (it is assumed that one person can buy only one product; in other words $P_0 \leq N$);

P_1 is the number of people in the market ($P_1 \leq N$) with average welfare;

P_2 is the number of people (critical level $P_2 \leq N$ and $P_1 \rightarrow P_2$), whose welfare has reached the limit of enrichment opportunities in this group of buyers;

P_3 are buyers who know that a given product is on the market ($P_3 \leq N$).

For convenience of research, we will rewrite the system of equations (1.8), (1.7), (1.9), and (1.5) in matrix form.

Let $\mathbf{P} = (P_0, P_1, P_2, P_3)^T \in \mathbb{R}^4$ and $\mathbf{Q} = (P_1, P_1 - P_2, P_2 - P_3, P_3)^T \in \mathbb{R}^4$.

Consider the system

$$\dot{\mathbf{P}}(t) = \mathbf{A} \cdot \mathbf{P}(t) + \epsilon \mu_0 P_3(t) \cdot \mathbf{B} \cdot \mathbf{Q}(t), \quad (2.1)$$

where

$$\mathbf{A} = \begin{pmatrix} -\lambda & \mu_0 & 0 & 0 \\ \lambda & -(\lambda + \mu_0) & \mu_0 & 0 \\ 0 & \lambda & -(\lambda + \mu_0) & \mu_0 \\ 0 & 0 & \lambda & -\mu_0 \end{pmatrix} \in \mathbb{R}^{4 \times 4},$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathbb{R}^{4 \times 4}.$$

Lemma 2.1. *Let $\lambda \geq 0$, $\mu_0 \geq 0$, $\lambda + \mu_0 > 0$, and $\epsilon \geq 0$. Then for any $P_0(0) \geq 0, P_1(0) \geq 0, P_2(0) \geq 0$, and $P_3(0) \geq 0$ the coordinates of solution $\mathbf{P}(t) = (P_0(t), P_1(t), P_2(t), P_3(t))^T$ of system (2.1) are nonnegative.*

Proof. We will consider the following cases:

(a) There exists the time $t_3 > 0$ such that $P_3(t_3) = 0$, $P_0(t_3) \geq 0$, $P_1(t_3) \geq 0$, $P_2(t_3) \geq 0$, and if $0 < t_{3s} < t_3$, then $P_3(t_{3s}) > 0$; thus, from the 4th equation of system (2.1) we get $\dot{P}_3(t_3) = \lambda P_2(t_3) \geq 0$. This means that for arbitrarily small $\Delta_3 > 0$, we have $P_3(t_3 + \Delta_3) > 0$.

(b) There exists the time $t_0 > 0$ such that $P_0(t_0) = 0$, $P_1(t_0) \geq 0$, $P_2(t_0) \geq 0$, $P_3(t_0) \geq 0$, and if $0 < t_{0s} < t_0$, then $P_0(t_{0s}) > 0$; thus, from the 1th equation

of system (2.1) we get $\dot{P}_0(t_0) = \mu_0 P_1(t_0) + \epsilon \mu_0 P_1(t_0) P_3(t_0) \geq 0$. This means that for arbitrarily small $\Delta_0 > 0$, we have $P_0(t_0 + \Delta_0) > 0$.

(c) There exists the time $t_1 > 0$ such that $P_1(t_1) = 0$, $P_0(t_1) \geq 0$, $P_2(t_1) \geq 0$, $P_3(t_1) \geq 0$, and if $0 < t_{1s} < t_1$, then $P_1(t_{1s}) > 0$; thus, from the 2th equation of system (2.1) we get $\dot{P}_1(t_1) = \lambda P_0(t_1) + \mu_0 P_2(t_1) + \epsilon \mu_0 P_2(t_1) P_3(t_1) \geq 0$. This means that for arbitrarily small $\Delta_1 > 0$, we have $P_1(t_1 + \Delta_1) > 0$.

(d) There exists the time $t_2 > 0$ such that $P_2(t_2) = 0$, $P_0(t_2) \geq 0$, $P_1(t_2) \geq 0$, $P_3(t_2) \geq 0$, and if $0 < t_{2s} < t_2$, then $P_2(t_{2s}) > 0$; thus, from the 3th equation of system (2.1) we get $\dot{P}_2(t_2) = \lambda P_1(t_2) + \mu_0 P_3(t_2) + \epsilon \mu_0 P_3^2(t_2) \geq 0$. This means that for arbitrarily small $\Delta_2 > 0$, we have $P_2(t_2 + \Delta_2) > 0$.

As a result, we come to the conclusion that under the conditions of Lemma 2.1 there doesn't exist the solution $\mathbf{P}(t)$ of system (2.1) with negative coordinates. The proof of Lemma 2.1 is complete. \square

Let us sum up all the equations of system (2.1). Then we get $\forall t \geq 0$ $P_0(t) + P_1(t) + P_2(t) + P_3(t) = C = \text{const}$, where $C \leq 4N$. This result allows us to simplify further research.

We will assume that $C = 1$. Then, instead of variables P_i , we will deal with relative variables $P_i/(4N)$. To simplify further presentation, we will again use the notation P_i for relative variables; $i = 0, \dots, 3$.

Important Assumption:

$$P_0(0) + P_1(0) + P_2(0) + P_3(0) \leq 1. \quad (2.2)$$

The following obvious lemma is a trivial corollary of Lemma 2.1.

Lemma 2.2. *Under the conditions of Lemma 2.1 and conditions (2.2), all coordinates of vector $\mathbf{P}(t)$ are bounded: $0 \leq P_i(t) \leq 1, i = 0, \dots, 3$.*

Without loss of generality we can assume that $P_0(0) + \dots + P_3(0) = 1$. From here it follows that $P_3 = 1 - P_0 - P_1 - P_2$. Using the last relation we reduce the order of system (2.1) by one. Then we have

$$\dot{\mathbf{P}}_1(t) = \mathbf{A}_1 \cdot \mathbf{P}_1(t) + \epsilon \mu_0 \cdot (P_0(t) + P_1(t) + P_2(t)) \cdot \mathbf{Q}_1(t) + \mu_0(1 + \epsilon) \cdot \mathbf{C}_1, \quad (2.3)$$

where

$$\mathbf{P}_1 = (P_0, P_1, P_2)^T \in \mathbb{R}^3, \quad \mathbf{C}_1 = (0, 0, 1)^T \in \mathbb{R}^3,$$

and

$$\mathbf{Q}_1 = (-P_1, P_1 - P_2, P_0 + P_1 + 2P_2)^T \in \mathbb{R}^3;$$

$$\mathbf{A}_1 = \begin{pmatrix} -\lambda & \mu_0(1 + \epsilon) & 0 \\ \lambda & -(\lambda + \mu_0 + \epsilon \mu_0) & \mu_0(1 + \epsilon) \\ -\mu_0(1 + 2\epsilon) & \lambda - \mu_0(1 + 2\epsilon) & -(\lambda + 2\mu_0 + 3\epsilon \mu_0) \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$

We introduce the vector norm:

$$\forall \mathbf{u} = (u_1, \dots, u_n)^T \in \mathbb{R}^n, \quad \|\mathbf{u}\| = \max_{1 \leq i \leq n} |u_i|.$$

Theorem 2.1. *Assume that the conditions of Lemma 2.2 are satisfied. Then system (2.3) has one stable equilibrium point in the hypercube*

$$\mathbb{N}^3 := \{0 \leq P_i \leq 1; i = 0, \dots, 2\} \subset \mathbb{R}^3.$$

Proof. We need to show that all solutions of system (2.3) starting from the hypercube \mathbb{N}^3 are stable. In addition, these solutions are attracted to a single equilibrium point, which is located in the specified hypercube.

Let's consider two cases.

2.1. The case $\epsilon = 0$

Taking into account the last remark, we get is (2.3) the line system

$$\dot{\mathbf{P}}_1(t) = \mathbf{A}_1 \cdot \mathbf{P}_1(t) + \mu_0 \cdot \mathbf{C}_1, \quad (2.4)$$

where

$$\mathbf{A}_1|_{\epsilon=0} = \begin{pmatrix} -\lambda & \mu_0 & 0 \\ \lambda & -(\lambda + \mu_0) & \mu_0 \\ -\mu_0 & \lambda - \mu_0 & -(\lambda + 2\mu_0) \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$

Let us show that in this case the matrix $\mathbf{A}_1|_{\epsilon=0}$ is Gurwitz [6].

Obviously, the unique equilibrium point of system (2.4) is determined from the following system of equations:

$$\begin{cases} -\lambda P_0 + \mu_0 P_1 = 0, \\ \lambda P_0 - (\lambda + \mu_0) P_1 + \mu_0 P_2 = 0, \\ -\mu_0 P_0 + (\lambda - \mu_0) P_1 - (\lambda + 2\mu_0) P_2 + \mu_0 = 0. \end{cases} \quad (2.5)$$

The solution of linear algebraic equations system (2.5) looks like this

$$P_{0e} = \frac{\mu_0^3}{(\lambda + \mu_0)(\mu_0^2 + \lambda_0^2)}, P_{1e} = \frac{\mu_0^2 \lambda}{(\lambda + \mu_0)(\mu_0^2 + \lambda_0^2)}, P_{2e} = \frac{\mu_0 \lambda^2}{(\lambda + \mu_0)(\mu_0^2 + \lambda_0^2)}.$$

Obviously, if $\lambda \geq 0, \mu_0 \geq 0$, and $\lambda + \mu_0 > 0$, then $\mathbf{P}_{1e} = (P_{0e}, P_{1e}, P_{2e})^T \in \mathbb{N}^3$. Note that from the last relations we can obtain the value of coordinate P_{3e} for the equilibrium point of system (2.1):

$$P_{3e} = 1 - P_{0e} - P_{1e} - P_{2e} = \frac{\lambda^3}{(\lambda + \mu_0)(\mu_0^2 + \lambda_0^2)} \leq 1.$$

Now let's calculate the characteristic polynomial of matrix $\mathbf{A}_1|_{\epsilon=0}$. Then we have

$$\begin{aligned} \det(\delta I_3 - \mathbf{A}_1|_{\epsilon=0}) &= \delta^3 + a_1 \delta^2 + a_2 \delta + a_3 \\ &= \delta^3 + 3(\lambda + \mu_0) \delta^2 + (3\lambda^2 + 3\mu_0^2 + 4\lambda\mu_0) \delta + (\lambda + \mu_0)(\lambda^2 + \mu_0^2). \end{aligned} \quad (2.6)$$

To ensure stability, we use the Hurwitz criterion [6]:

$$a_1 > 0, a_2 > 0, a_3 > 0, a_1 a_2 > a_3. \quad (2.7)$$

Note that the form $a_2 = 3\lambda^2 + 3\mu_0^2 + 4\lambda\mu_0$ is positive definite. Further, the condition $a_1 a_2 > a_3$ leads to the cubic inequality:

$$a_1 a_2 - a_3 = (\lambda + \mu_0)(8\lambda^2 + 8\mu_0^2 + 12\lambda\mu_0) > 0,$$

where the form $8\lambda^2 + 8\mu_0^2 + 12\lambda\mu_0$ is positive definite. Therefore, if $(\lambda + \mu_0) > 0$, then the Hurwitz criterion is valid.

According to the well-known theorem of polynomial algebra (Gua's theorem), if condition $a_k^2 \leq a_{k-1} a_{k+1}$, $k = 1, 2$, is satisfied for at least one triple of coefficients $(a_0, a_1, a_2$ or $a_1, a_2, a_3)$, then the polynomial $\delta^3 + a_1 \delta^2 + a_2 \delta + a_3$ has at least one pair of complex roots. We have

$$\begin{aligned} (3\lambda + 3\mu_0)^2 &> (3\lambda^2 + 3\mu_0^2 + 4\lambda\mu_0) \\ \Rightarrow \forall \lambda > 0, \forall \mu_0 > 0 \quad 6\lambda^2 + 6\mu_0^2 + 14\lambda\mu_0 &> 0 \end{aligned}$$

and

$$\begin{aligned} (3\lambda^2 + 3\mu_0^2 + 4\lambda\mu_0)^2 &> 3(\lambda + \mu_0)^2(\lambda^2 + \mu_0^2) \\ \Rightarrow \forall \lambda > 0, \forall \mu_0 > 0 \quad 6(\lambda^2 + \mu_0^2)^2 + \lambda\mu_0(30\lambda^2 + 30\mu_0^2 + 28\lambda\mu_0) &> 0. \end{aligned}$$

Consequently, Gua's theorem does not hold for any of the triples of coefficients of the polynomial. From here it follows that all roots of polynomial $\delta^3 + a_1 \delta^2 + a_2 \delta + a_3$ are real and negative.

Thus, there is a neighborhood of the equilibrium point $\mathbf{P}_{1e} \in \mathbb{N}^3$ in which system (2.4) is stable. This equilibrium point is unique in \mathbb{N}^3 .

2.2. The case $\epsilon \neq 0$

(b1) According to Lemma 2.1, all solutions of system (2.4) are bounded. Note that to prove this fact it was not required that the matrix \mathbf{A}_1 be Hurwitz.

Therefore, now we will consider the opposite case: the matrix \mathbf{A}_1 is not Hurwitz.

Let's transform equation (2.3) into the following integral equation

$$\begin{aligned} \mathbf{P}_1(t) &= \exp(\mathbf{A}_1 t) \mathbf{P}_1(0) \\ &+ \int_0^t \exp(\mathbf{A}_1(t - \tau)) (\epsilon \mu_0 \cdot (P_0(t) + P_1(t) + P_2(t)) \cdot \mathbf{Q}_1(t)) d\tau \\ &+ \int_0^t \exp(\mathbf{A}_1(t - \tau)) (1 + \epsilon) \mu_0 \cdot \mathbf{C}_1 d\tau. \end{aligned} \quad (2.8)$$

From here it follows that

$$\begin{aligned} \|\mathbf{P}_1(t)\| &\leq \|\exp(\mathbf{A}_1 t)\| \|\mathbf{P}_1(0)\| \\ &+ \epsilon \mu_0 \int_0^t \|\exp(\mathbf{A}_1(t-\tau))\| \|(P_0(t) + P_1(t) + P_2(t)) \cdot \mathbf{Q}_1(t) + \mathbf{C}_1\| d\tau \\ &+ \mu_0 \int_0^t \|\exp(\mathbf{A}_1(t-\tau))\| \|\mathbf{C}_1\| d\tau. \end{aligned} \quad (2.9)$$

By virtue of condition (2.2) we get

$$\|(P_0(t) + P_1(t) + P_2(t)) \cdot \mathbf{Q}_1(t) + \mathbf{C}_1\| \leq 3.$$

Now let's use the known result of matrix analysis: $\|\exp(\mathbf{A}_1 t)\| \leq \beta \exp(\alpha t)$, $\beta > 0$ is valid; here $\alpha > 0$ is the maximum real part of eigenvalues of the matrix \mathbf{A}_1 .

Thus, inequality (2.9) can be rewritten in the form

$$\begin{aligned} \|\mathbf{P}_1(t)\| &\leq \beta \exp(\alpha t) \|\mathbf{P}_1(0)\| + \beta \mu_0 (3\epsilon + 1) \int_0^t \exp(\alpha(t-\tau)) d\tau \\ &\leq \beta \exp(\alpha t) \|\mathbf{P}_1(0)\| + \frac{\beta \mu_0 (3\epsilon + 1)}{\alpha} (\exp(\alpha t) - 1). \end{aligned} \quad (2.10)$$

Obviously, for $t \rightarrow \infty$ for any solution $\mathbf{P}_1(t)$ of equation (2.3) we get $\|\mathbf{P}_1(t)\| \rightarrow \infty$. Thus, if $\alpha > 0$, then for sufficiently large $t > 0$ the solution $\mathbf{P}_1(t) \notin \mathbb{N}^3$. Therefore, in order that $\mathbf{P}_1(t) \in \mathbb{N}^3$ the matrix \mathbf{A}_1 must be Hurwitz.

(b2) We calculate the coefficients of the characteristic polynomial

$$\delta^3 + a_1 \delta^2 + a_2 \delta + a_3$$

of the matrix \mathbf{A}_1 :

$$\begin{aligned} a_1 &= 3\lambda + 3\mu_0 + 4\epsilon\mu_0, a_2 \\ &= 3\lambda^2 + 3\mu_0^2 + 4\lambda\mu_0 + 6\lambda\mu_0\epsilon + 8\epsilon\mu_0^2 + 5\epsilon^2\mu_0^2, \\ a_3 &= \mu_0^3 + 3\lambda\epsilon\mu_0^2 + 5\epsilon^2\mu_0^3 + 2\epsilon^3\mu_0^3 + \lambda^3 + \lambda^2\mu_0 \\ &+ 2\lambda\epsilon^2\mu_0^2 + \lambda\mu_0^2 + 4\epsilon\mu_0^3 + 2\lambda^2\epsilon\mu_0. \end{aligned}$$

Now let's use the Hurwitz criterion (2.7). Since all coefficients a_1, a_2, a_3 are positive, we only need to check one condition: $a_1 a_2 - a_3 > 0$. We have

$$\begin{aligned} a_1 a_2 - a_3 &= (3\lambda + 3\mu_0 + 4\epsilon\mu_0) \\ &\times (3\lambda^2 + 3\mu_0^2 + 4\lambda\mu_0 + 6\lambda\mu_0\epsilon + 8\epsilon\mu_0^2 + 5\epsilon^2\mu_0^2) \\ &- (\mu_0^3 + 3\lambda\epsilon\mu_0^2 + 5\epsilon^2\mu_0^3 + 2\epsilon^3\mu_0^3 + \lambda^3) \\ &- (\lambda^2\mu_0 + 2\lambda\epsilon^2\mu_0^2 + \lambda\mu_0^2 + 4\epsilon\mu_0^3 + 2\lambda^2\epsilon\mu_0) \\ &= (\lambda + \mu_0)(8\lambda^2 + 8\mu_0^2 + 12\lambda\mu_0) + 18\epsilon^3\mu_0^3 + \epsilon^2\mu_0^2(42\mu_0 + 37\lambda) \\ &+ \epsilon\mu_0(28\lambda^2 + 32\mu_0^2 + 58\lambda\mu_0) > 0. \end{aligned}$$

If $\lambda \geq 0$, $\mu_0 \geq 0$, $\lambda + \mu_0 > 0$, and $\epsilon \geq 0$, then the last inequality is always true. Therefore, the matrix \mathbf{A}_1 (and for $\epsilon \neq 0$) is indeed Hurwitz.

(b3) It remains to show that the stable equilibrium point $\mathbf{P}_{1e} \in \mathbb{N}^3$ is unique.

Let's put $\mathbf{H}_1 = \mathbf{P}_1 \in \mathbb{R}^3$ and $\mathbf{Q}_1 = (-H_1, H_1 - H_2, H_0 + H_1 + 2H_2)^T \in \mathbb{R}^3$. Along with the integral equation (2.8), we will consider the same integral equation

$$\begin{aligned} \mathbf{H}_1(t) &= \exp(\mathbf{A}_1 t) \mathbf{H}_1(0) \\ &+ \int_0^t \exp(\mathbf{A}_1(t-\tau)) (\epsilon \mu_0 \cdot (H_0(t) + H_1(t) + H_2(t)) \cdot \mathbf{Q}_1(t)) d\tau \\ &+ \int_0^t \exp(\mathbf{A}_1(t-\tau)) (1 + \epsilon) \mu_0 \cdot \mathbf{C}_1 d\tau, \end{aligned} \quad (2.11)$$

but with different initial data $H_i(0) \in \mathbb{N}^3, i = 0, \dots, 3$.

Suppose that in the hypercube \mathbb{N}^3 there are two different equilibrium points: \mathbf{P}_{1e} and \mathbf{H}_{1e} . Then there is a surface $\mathbb{G} \subset \mathbb{N}^3$ dividing the hypercube into two regions \mathbb{N}_1^3 and \mathbb{N}_2^3 such that $\mathbb{N}_1^3 \cup \mathbb{G} \cup \mathbb{N}_2^3 = \mathbb{N}^3$, $\mathbb{N}_1^3 \cap \mathbb{N}_2^3 = \emptyset$. Thus, we have $\mathbf{P}_{1e} \in \mathbb{N}_1^3$ and $\mathbf{H}_{1e} \in \mathbb{N}_2^3$.

We estimate the magnitudes $\lim_{t \rightarrow \infty} \|\mathbf{P}_1(t)\|$ and $\lim_{t \rightarrow \infty} \|\mathbf{H}_1(t)\|$. For this purpose we will use equations (2.8) and (2.11):

$$\begin{aligned} \|\mathbf{P}_1(t)\| &\leq \|e^{\mathbf{A}_1 t} \mathbf{P}_1(0)\| \\ &+ \left\| \int_0^t e^{\mathbf{A}_1(t-\tau)} (\epsilon \mu_0 (P_0(t) + P_1(t) + P_2(t)) \cdot \mathbf{Q}_1(\mathbf{P}_1(t))) d\tau \right. \\ &\quad \left. + \int_0^t e^{\mathbf{A}_1(t-\tau)} (1 + \epsilon) \mu_0 \cdot \mathbf{C}_1 d\tau \right\|; \\ \|\mathbf{H}_1(t)\| &\leq \|e^{\mathbf{A}_1 t} \mathbf{H}_1(0)\| \\ &+ \left\| \int_0^t e^{\mathbf{A}_1(t-\tau)} (\epsilon \mu_0 (H_0(t) + H_1(t) + H_2(t)) \cdot \mathbf{Q}_1(\mathbf{H}_1(t))) d\tau \right. \\ &\quad \left. + \int_0^t e^{\mathbf{A}_1(t-\tau)} (1 + \epsilon) \mu_0 \cdot \mathbf{C}_1 d\tau \right\|. \end{aligned}$$

We introduce the designations

$$\forall \mathbf{P}_1(0) \in \mathbb{N}_1^3 \quad \|(P_0(t) + P_1(t) + P_2(t)) \cdot \mathbf{Q}_1(\mathbf{P}_1(t)) + \mathbf{C}_1\| = \xi \leq 3$$

and

$$\forall \mathbf{H}_1(0) \in \mathbb{N}_2^3 \quad \|(H_0(t) + H_1(t) + H_2(t)) \cdot \mathbf{Q}_1(\mathbf{H}_1(t)) + \mathbf{C}_1\| = \psi \leq 3.$$

Now we use the notation adopted in inequality (2.10). Then we get

$$\begin{aligned}\|\mathbf{P}_1(t)\| &\leq \beta \exp(-\alpha t)\|\mathbf{P}_1(0)\| + \frac{\beta\mu_0(\xi\epsilon + 1)}{\alpha}(1 - \exp(-\alpha t)) \\ &\leq \frac{\beta\mu_0(3\epsilon + 1)}{\alpha}; \\ \|\mathbf{H}_1(t)\| &\leq \beta \exp(-\alpha t)\|\mathbf{H}_1(0)\| + \frac{\beta\mu_0(\psi\epsilon + 1)}{\alpha}(1 - \exp(-\alpha t)) \\ &\leq \frac{\beta\mu_0(3\epsilon + 1)}{\alpha}.\end{aligned}$$

(Here, we take into account the fact that the real part of any eigenvalue of the Hurwitz matrix is negative. Therefore, we have $-\alpha < 0$.)

Thus, we have

$$\lim_{t \rightarrow \infty} \|\mathbf{P}_1(t) - \mathbf{H}_1(t)\| \geq \lim_{t \rightarrow \infty} \left| \|\mathbf{P}_1(t)\| - \|\mathbf{H}_1(t)\| \right| \geq 0.$$

Let us assume that at the moment $t = 0$ the initial conditions for solutions $\mathbf{P}_1(t)$ and $\mathbf{H}_1(t)$ are taken at some point L on the boundary \mathbb{G} . In other words $\mathbf{P}_{1L}(0) = \mathbf{H}_{1L}(0)$.

We introduce a function $V(\mathbf{P}_1)$ on the set \mathbb{N}^3 such that $\forall \mathbf{P}_1 \in \mathbb{N}^3$ we get $\dot{V}_t(\mathbf{P}_1) \leq 0$ and $\forall L \in \mathbb{G}$ we also have $\dot{V}_t(L) = 0$. (The derivative of function $V(\mathbf{P}_1)$ is defined by virtue of system (2.3): $\dot{V}_t(\mathbf{P}_1) = (\partial V(\mathbf{P}_1)/\partial \mathbf{P}_1) \cdot \dot{\mathbf{P}}_1(t)$.) Then from condition $\forall Q \in (\mathbb{N}^3 - \mathbb{G})$ it follows that $\dot{V}_t(Q) < 0$.

Obviously, if the point $L \in \mathbb{G}$, then at this point the matrix \mathbf{A}_1 is not Hurwitz. (Along the surface \mathbb{G} we have either $\dot{V}_t(\mathbb{G}) = 0$ or $\dot{V}_t(\mathbb{G})$ doesn't exist.) We obtain a contradiction with the assumption that at any points of the hypercube \mathbb{N}^3 the matrix \mathbf{A}_1 must be Hurwitz. Therefore, the following relations must be true: $\mathbb{G} = \emptyset$, $\mathbb{N}_1^3 = \mathbb{N}_2^3 = \mathbb{N}^3$, and $\lim_{t \rightarrow \infty} \|\mathbf{P}_1(t)\| = \lim_{t \rightarrow \infty} \|\mathbf{H}_1(t)\|$. This means that point \mathbf{P}_{1e} is unique. The proof of Theorem 2.1 is complete. \square

The results of application of Theorem 2.1 were demonstrated on Fig.2.1.

3. Generalization

Theorem 2.1 can be generalized to the n -dimensional case, where $n > 3$.

Let's consider a more complex situation than that described in model (2.1).

We will assume that the variables $P_0(t)$ and $P_n(t)$ (if $n = 3$, then $P_n(t) = P_3(t)$) have the same meaning as in system (2.1). As for the variables $P_1(t)$ and $P_2(t)$, we will now assume that there are $m > 1$ groups, each of which consists of two variables: $(P_1, P_2), (P_3, P_4), \dots, (P_{2m-1}, P_{2m}) = (P_{n-2}, P_{n-1})$. (From here it follows that $n = 2m + 1$.)

Note that the economic meaning of the variables P_{2i-1} and P_{2i} is the same as the variables P_1 and P_2 in system (2.1); $i = 1, \dots, m$. The difference (in the new sales model) is in structuring buyers with various welfare, which was carried out more carefully than it was previously.

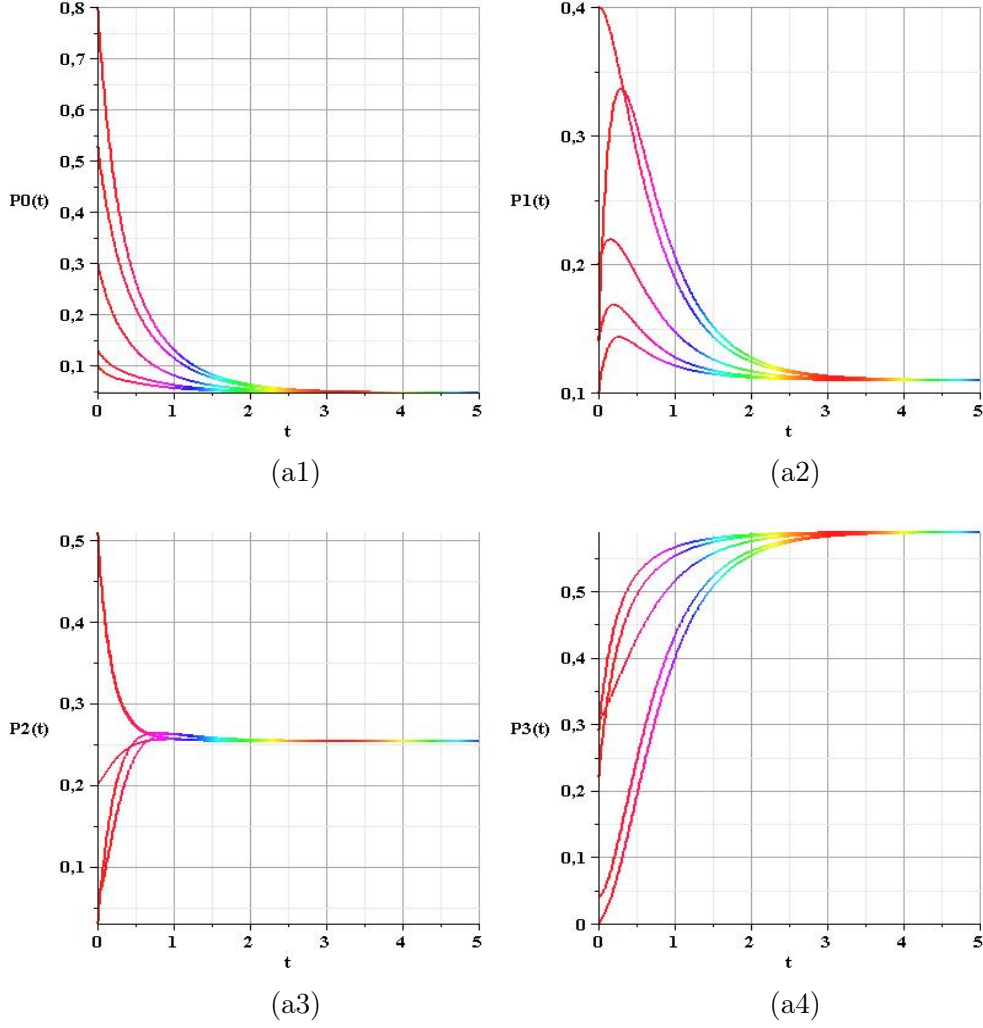


Fig. 2.1. Graphs of coordinates of the phase trajectory $\mathbf{P}(t)$ of system (2.1) for various initial data: (a1) $P_0(t)$, (a2) $P_1(t)$, (a3) $P_2(t)$, (a4) $P_3(t)$. For any initial data determined by Theorem 2.1, the trajectory $\mathbf{P}(t)$ tends to the limit point $\mathbf{P}_e = (0.05, 0.12, 0.26, 0.57)^T$. Here $\lambda = 3, \mu_0 = 1, \epsilon = 0.5$.

Let's introduce the following matrix

$$\mathbf{A} = \begin{pmatrix} -\lambda & \mu_0 & 0 & \dots & 0 & 0 \\ \lambda & -(\lambda + \mu_0) & \mu_0 & \dots & 0 & 0 \\ 0 & \lambda & -(\lambda + \mu_0) & \dots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & \mu_0 & 0 \\ 0 & 0 & 0 & \dots & -(\lambda + \mu_0) & \mu_0 \\ 0 & 0 & 0 & \dots & \lambda & -\mu_0 \end{pmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}.$$

By symbols $\mathbf{P} = (P_0, P_1, P_2, \dots, P_n)^T \in \mathbb{R}^{n+1}$ and $\mathbf{Q} = (P_1, P_2 - P_1, P_3 - P_2, \dots, P_{n-1} - P_{n-2}, P_n - P_{n-1}, -P_n)^T \in \mathbb{R}^{n+1}$, we denote vectors of \mathbb{R}^{n+1} . Consider the system of differential equations

$$\dot{\mathbf{P}}(t) = \mathbf{A} \cdot \mathbf{P}(t) + \epsilon \mu_0 P_n(t) \cdot \mathbf{Q}(t), \lambda \geq 0, \mu_0 \geq 0, \epsilon \geq 0, \quad (3.1)$$

with initial values $\mathbf{P}(0) \in \mathbb{R}^n$.

Let's find the sum Σ of all rows of matrix \mathbf{A} . Then we get $\Sigma = 0$. This means that matrix \mathbf{A} has one zero eigenvalue. From here it follows that the characteristic polynomial of matrix \mathbf{A} should look like this

$$\det(\delta \cdot I_n - \mathbf{A}) = \delta \cdot (\delta^n + a_1 \delta^{n-1} + \dots + a_n) \equiv \delta \cdot \phi(\delta).$$

Here $I_n \in \mathbb{R}^{(n+1) \times (n+1)}$ is the identity matrix and $\phi(\delta)$ is a polynomial of degree n .

Let $\mathbb{N}^{n+1} := \{0 \leq P_i \leq 1; i = 0, \dots, n\}$, be a hypercube in \mathbb{R}^{n+1} with edge equal to 1.

Theorem 3.1. *Assume that for system (3.1) the following conditions are fulfilled:*

(a1) $\mathbf{P}(0) \in \mathbb{N}^{n+1}$ and $P_0(0) + P_1(0) + \dots + P_n(0) = 1$;

(a2) $\lambda + \mu_0 > 0$;

(a3) $\phi(\delta)$ is the Hurwitz polynomial.

Then system (3.1) has one stable equilibrium point $\mathbf{P}_e \in \mathbb{N}^{n+1}$ and

$$\lim_{t \rightarrow \infty} \mathbf{P}(t) = \mathbf{P}_e.$$

Proof. Let us first assume that conditions (a1) and (a2) are satisfied. Since each row of the matrix \mathbf{A} contains no more than three elements, the procedure for proving the statements of Lemmas 2.1 and 2.2 can be carried over verbatim to the n -dimensional case.

As to condition (a3), its role in the proof of the theorem will be demonstrated below.

Let's represent the vector \mathbf{Q} in the following form

$$\mathbf{Q} = D \cdot \mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_{n-1} \\ P_n \end{pmatrix} \in \mathbb{R}^{(n+1)},$$

where $D \in \mathbb{R}^{(n+1) \times (n+1)}$.

Now we will find the equilibrium points of system (3.1). To do this, it is necessary to solve the system of equations

$$(\mathbf{A} + \epsilon\mu_0 P_n D)\mathbf{P} = 0. \quad (3.2)$$

(It is clear that system (3.2) has a trivial solution $\mathbf{P} = 0$.)

We sum up all the equations of system (3.1). Then we get $\dot{P}_0(t) + \dot{P}_1(t) + \dots + \dot{P}_n(t) = 0$. Integrating the last relation we find $P_0 + P_1 + \dots + P_n = C$. In what follows we assume that $C = 1$.

Thus, we need to solve system (3.2), in which the last equation will be replaced by equation

$$P_0 + P_1 + \dots + P_n = 1. \quad (3.3)$$

By sequentially solving the equations of system (3.2) we arrive at the following relations

$$P_0 = \frac{\mu_0(1 + \epsilon P_n)P_1}{\lambda}, P_1 = \frac{\mu_0(1 + \epsilon P_n)P_2}{\lambda}, \dots, P_{n-1} = \frac{\mu_0(1 + \epsilon P_n)P_n}{\lambda}. \quad (3.4)$$

Now we use relation (3.3). Then we obtain the following polynomial equation for P_n :

$$P_n \cdot \left[\frac{\mu_0^n (1 + \epsilon P_n)^n}{\lambda^n} + \dots + \frac{\mu_0(1 + \epsilon P_n)}{\lambda} + 1 \right] - 1 = 0. \quad (3.5)$$

All coefficients λ , ϵ , and μ_0 in equation (3.5) (except the last one) are positive. Therefore, the number of sign changes for polynomial (3.5) is equal to one. Then, according to Descartes' well-known theorem, equation (3.5) has a single positive root $P_n^* > 0$.

Now you can use formulas (3.4) to calculate all the remaining coordinates of the equilibrium point. All these coordinates $P_0^*, P_1^*, \dots, P_{n-1}^*$ are positive and satisfy condition (3.3).

Suppose that the polynomial $\phi(\delta)$ is not Hurwitz. Then, as shown in the proof of Theorem 2.1, the trajectory of system (2.3) leaves the hypercube \mathbb{N}^3 . The exclusion of such a situation is the main reason for introducing condition (a3). The proof of Theorem 3.1 is complete. \square

Note that an important difference between Theorem 3.1 and Theorem 2.1 is that, unlike Theorem 2.1, we cannot derive condition (a3) from conditions (a1) and (a2). Therefore, in Theorem 3.1 condition (a3) was simply postulated.

4. Changing assumption (2.2)

Models (2.1) and (3.1), described in the previous sections, were based on one important circumstance: under the conditions of Lemmas 2.1 and 2.2, system (2.1) (and (3.1)) had only one equilibrium point. In this case, it is impossible to perform a complete analysis of the dynamics of the sales market.

The situation changes if we assume that groups of buyers, the dynamics of which are described by the functions $P_0(t), \dots, P_3(t)$, can intersect. In this case,

condition (2.2) is replaced by condition $P_0(t) + \dots + P_3(t) > 1$. In other words, the functions $P_0(t), \dots, P_3(t)$ must satisfy condition $P_0(t) + \dots + P_3(t) \leq 4N$. Now, if we go to relative variables $P_0/N, \dots, P_3/N$, then condition (2.2) will be replaced by the condition

$$P_0(0) + \dots + P_3(0) \leq 4, \quad (4.1)$$

where $P_0(0) \geq 0, \dots, P_3(0) \geq 0$.

Note that under condition (4.1) of the previous results, only Lemma 2.1 remains valid. As for the Lemma (2.2), it should be reformulated as follows:

Lemma 4.1. *Under the conditions of Lemma 2.1 and conditions (4.1), all coordinates of vector $\mathbf{P}(t)$ are bounded: $0 \leq P_i(t) \leq 4, i = 0, \dots, 3$.*

Besides condition (4.1), the following condition

$$0 \leq P_0(0) \leq 1, \dots, 0 \leq P_3(0) \leq 1 \quad (4.2)$$

must also be taken into account.

Taking into account conditions (4.1) and (4.2) leads to the fact that, in contrast to condition (2.2), the behavior of the functions $P_0(t), \dots, P_3(t)$ becomes completely different. The difference is that the fulfillment of condition (4.2) imposes additional restrictions on the parameters λ, μ_0, ϵ in addition to those restrictions that are specified in Lemmas 2.1 and 4.1. In this work we will not write out these mathematical restrictions, but will limit ourselves only to some economic aspects.

The following graphs Fig.4.1, Fig.4.2, and Fig.4.3 show the change in the number of sales of a product depending on the initial conditions. (The parameters defining the structure of model (2.1), are taken equal to $\lambda = 3, \mu_0 = 3$, and $\epsilon = 0.5$; conditions (4.2) are valid.)

The visual analysis of graphs Fig.4.1, Fig.4.2, and Fig.4.3 allows us to draw the following conclusions:

1. The number of sales increases with increasing market saturation and intensifying advertising campaign (despite the decrease in average welfare) (see Fig.4.1).
2. The number of sales increases with increasing market saturation and increasing average welfare (despite the decrease in advertising product) (see Fig.4.2).
3. The number of sales increases with the growth of welfare and the volume of advertising product (see Fig.4.3).

If the conditions (2.2) are violated, but the conditions of the lemma 4.1 and conditions (4.2) are preserved, several equilibrium points may arise in system (2.1) (see Fig.4.4). In Fig.4.4 shows the projection of this straight line onto the plane

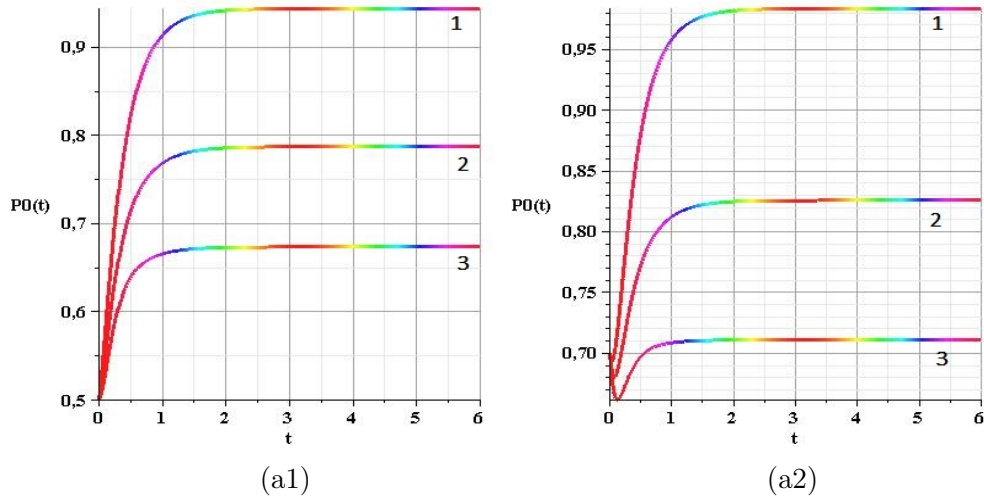


Fig. 4.1. Sales of goods on the market depending on the initial conditions: (a1) $P_0 = 0.5$, $P_1 = 0.5$, $P_2 = 1.0$, where curve 1 $P_3 = 0.8$, curve 2 $P_3 = 0.4$, curve 3 $P_3 = 0.1$; (a2) $P_0 = 0.7$, $P_1 = 0.4$, $P_2 = 1.0$, where curve 1 $P_3 = 0.8$, curve 2 $P_3 = 0.4$, curve 3 $P_1 = 0.1$.

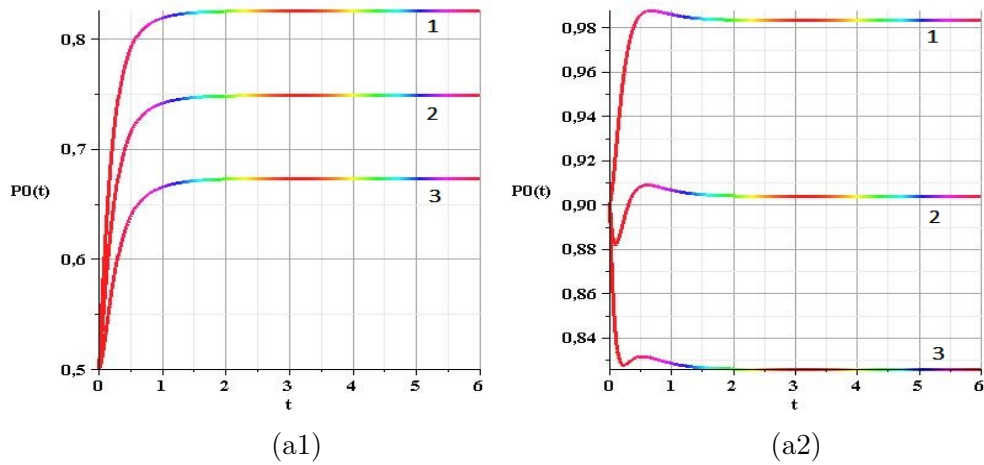


Fig. 4.2. Sales of goods on the market depending on the initial conditions: (a1) $P_0 = 0.5$, $P_2 = 1.0$, $P_3 = 0.1$, where curve 1 $P_1 = 0.9$, curve 2 $P_1 = 0.7$, curve 3 $P_1 = 0.5$; (a2) $P_0 = 0.9$, $P_2 = 1.0$, $P_3 = 0.1$, where curve 1 $P_1 = 0.9$, curve 2 $P_1 = 0.7$, curve 3 $P_1 = 0.5$.

$P_3 - P_0$ for parameters $\lambda = 3$, $\mu_0 = 3$, and $\epsilon = 0.5$ at different initial values. Moreover, as can be seen from Fig.4.4, these equilibrium points form a straight line in phase space.

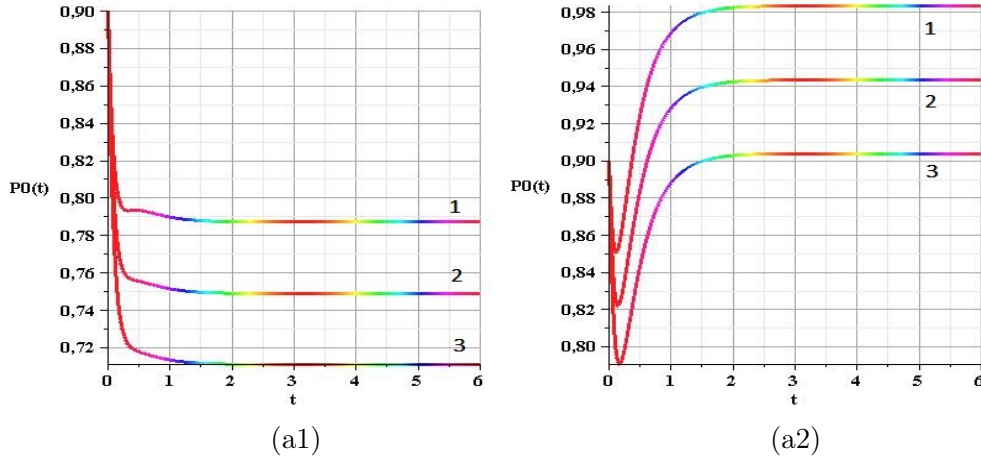


Fig. 4.3. Sales of goods on the market depending on the initial conditions: (a1) $P_0 = 0.9$, $P_2 = 1.0$, $P_3 = 0.1$, where curve 1 $P_1 = 0.4$, curve 2 $P_1 = 0.3$, curve 3 $P_1 = 0.2$; (a2) $P_0 = 0.9$, $P_2 = 1.0$, $P_3 = 0.6$, where curve 1 $P_1 = 0.4$, curve 2 $P_1 = 0.3$, curve 3 $P_2 = 0.2$.

5. Informal behavior of trajectories of system (2.1)

All results obtained in the previous sections have practical significance. However, there always remains a scientific interest in studying the behavior of trajectories of system (2.1), which goes beyond the scope prescribed by Theorem 2.1. This interest is dictated by the following question: are there such values of the parameters of system (2.1), at which the appearance of periodic trajectories (see [2, 3]) in the system is possible?

As shown in the proof of Theorem 2.1, if its conditions are satisfied, periodic trajectories in system (2.1) do not exist. However, when changing the conditions of Theorem 2.1, periodic trajectories may appear. Numerical experiments have shown that if parameters λ , μ_0 , and ϵ satisfy the constraints $\lambda > 0$, $\lambda + \mu_0 < 0$, $\epsilon < 0$, and the parameter $|\epsilon|$ is small enough, then limit cycles can be born in system (2.1) (see Fig.5.1 and Fig.5.2).

The statements of the last paragraph can be formulated more strictly in the form of the following theorem.

Theorem 5.1. *Assume that for system (2.1) the following conditions:*

(b1) $\lambda > 0$, $\lambda + \mu_0 \leq 0$, and $\epsilon < 0$;

(b3) *the matrix \mathbf{A} has a pair of complex conjugate eigenvalues with a positive real part, as well as one positive and one zero eigenvalues;*

(b3) $P_3(0) > 0$

are valid.

Then in system (2.1) for sufficiently small $|\epsilon| > 0$ there exists a limit cycle.

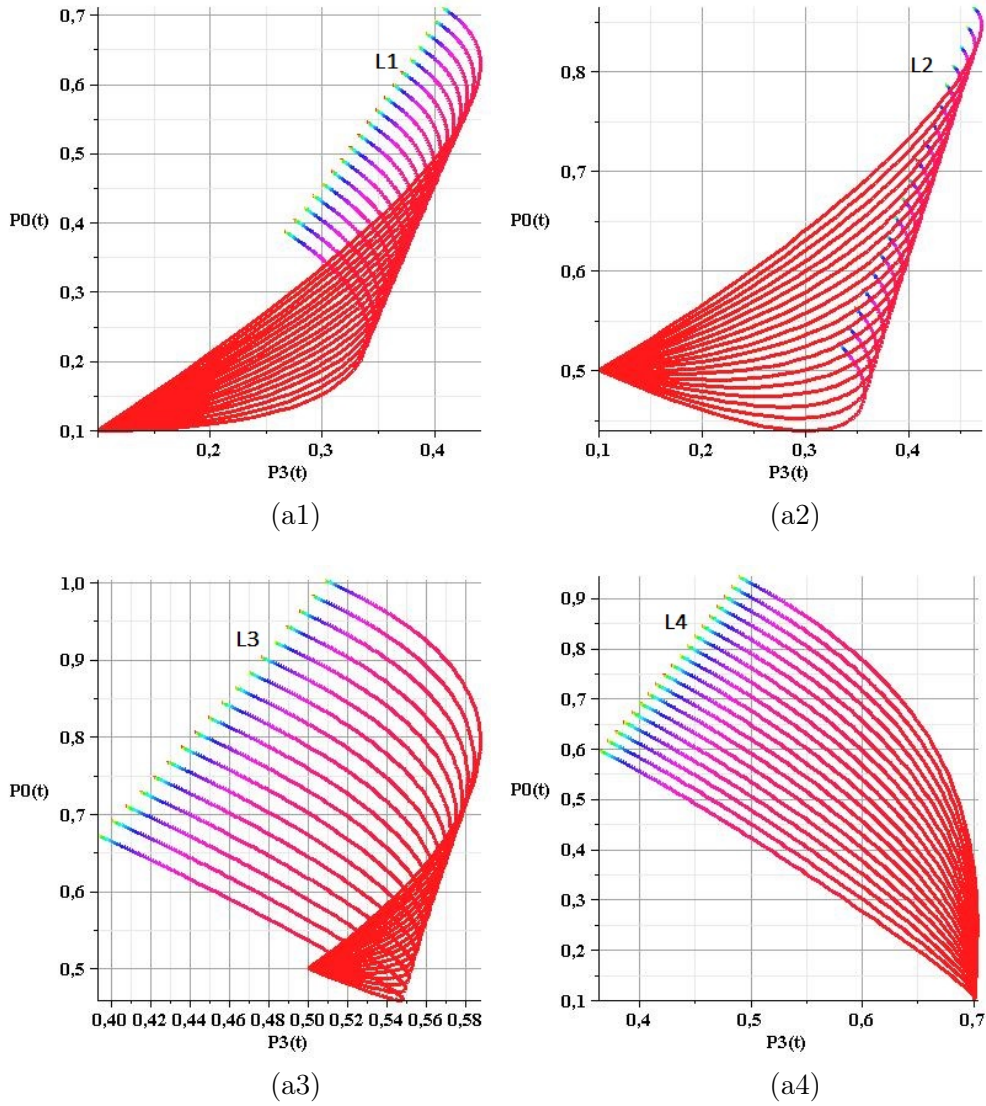


Fig. 4.4. Graphs of the dependence of the number of sales $P_0(t)$ on the volume of advertising $P_3(t)$ (under different initial conditions, the equilibrium points are also different): (a1) $P_0 = 0.1, P_3(0) = 0.1, P_2(0) = 1.0, P_1(0) = [0.1, 0, 15, \dots, 0.95, 1.0]$, L_1 is the line of equilibrium points; (a2) $P_0 = 0.5, P_3(0) = 0.1, P_2(0) = 1.0, P_1(0) = [0.1, 0, 15, \dots, 0.95, 1.0]$, L_2 is the line of equilibrium points; (a3) $P_0 = 0.5, P_3(0) = 0.5, P_2(0) = 1.0, P_1(0) = [0.1, 0, 15, \dots, 0.95, 1.0]$, L_3 is the line of equilibrium points; (a4) $P_0 = 0.1, P_3(0) = 0.7, P_2(0) = 1.0, P_1(0) = [0.1, 0, 15, \dots, 0.95, 1.0]$, L_4 is the line of equilibrium points. $L_1 - L_4$ are straight lines.

Proof. In what follows we will use equation (1.1) and its solution (1.2). In order that this use to be correct it is necessary to introduce the following redesignations of the parameters of equation (1.1): $\lambda \cdot A \rightarrow -\mu_0 > 0$, $\lambda \rightarrow \epsilon \cdot \mu_0 > 0$. In addition, it is necessary to do replacements $x \rightarrow t$ and $V(x) \rightarrow P_3(t)$.

Let's consider the 4th equation of of system (2.1) (this is equation (1.5)). If $P_2(0) \geq 0$, then according to the Comparison Principle [2,6]), we have $-\mu_0 P_3(t) - \epsilon \mu_0 P_3^2(t) \leq \lambda P_2(t) - \mu_0 P_3(t) - \epsilon \mu_0 P_3^2(t)$. Thus, if $P_3(0) \leq 0$, then from (1.2) it follows that solution $P_3(t) \leq 0$ has a singular point t^* such that $\lim_{t \rightarrow t^*} P_3(t) = -\infty$. Therefore, there must be $P_3(0) > 0$.

(a) Let us assume that $\epsilon = 0$, $P_0(0) + \dots + P_3(0) = 0$, and $\exists i \in \{0, \dots, 2\}$ such that $P_i(0) \neq 0$.

Under these assumptions, system (2.1) is transformed into system

$$\dot{\mathbf{P}}_1(t) = \mathbf{A}_{10} \cdot \mathbf{P}_1(t), \mathbf{P}_1(t) \in \mathbb{R}^3, \quad (5.1)$$

where all notations are the same as in system (2.4) and $\mathbf{A}_{10} = \mathbf{A}_1|_{\epsilon=0}$.

Let us calculate the characteristic polynomial of the matrix \mathbf{A}_{10} for $\lambda + \mu_0 = 0$. Then from formula (2.6) we have $\det(\delta I_3 - \mathbf{A}_{10}) = \delta^3 + 2\lambda^2 \delta$. The last equation has roots $\delta_{1,2} = \pm i\sqrt{2\lambda}$ and $\delta_3 = 0$.

By $\{\mathbf{s}_1, \mathbf{s}_2\} \subset \mathbb{R}^3$ denote the invariant under the operator \mathbf{A}_{10} $2D$ -subspace corresponding to the complex eigenvalues δ_1, δ_2 . Let also $\{\mathbf{s}_3\} \in \mathbb{R}^3$ be the eigenvector of the operator \mathbf{A}_{10} corresponding to the eigenvalue δ_3 .

Let's introduce the matrix $S = (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) \in \mathbb{R}^{3 \times 3}$. Since $\det S \neq 0$, then by changing variables $\mathbf{P}_1 = S\mathbf{W}$, we can transform system (5.1) into system

$$\dot{\mathbf{W}}(t) = S^{-1} \mathbf{A}_{10} S \cdot \mathbf{W}(t), \mathbf{W}(t) \in \mathbb{R}^3, \quad (5.2)$$

where the matrix $S^{-1} \mathbf{A}_{10} S$ has a quasi-diagonal structure: $\text{diag}(T, 0)$, $T \in \mathbb{R}^{2 \times 2}$.

System (5.2) has a unique nonzero eigenvector $(0, 0, \alpha \neq 0)^T \in \mathbb{R}^3$. The restriction of the operator $S^{-1} \mathbf{A}_{10} S$ to subspace $\{\mathbf{s}_1, \mathbf{s}_2\}$ also has a unique equilibrium point $\mathbf{0} \in \mathbb{R}^2$, which is the center [6,9]. Therefore, if we choose the initial vector $\mathbf{W}_0 = (w_{10}, w_{20}, 0)^T$, where $w_{10} \neq 0$ or $w_{20} \neq 0$, we will obtain a periodic solution to system (5.2).

Now, if we do the reverse change of variables $\mathbf{W} = S^{-1} \mathbf{P}_1$ and return to system (5.1), then we obtain the periodic solution of this system, which should start from point $\mathbf{P}_{10} = S \cdot \mathbf{W}_0 = \mathbf{P}_1(0) = (P_{10}(0), P_{11}(0), P_{12}(0))^T$.

(b) Now we assume that $\epsilon \neq 0$. Let us return to system (2.1) under the condition $P_0(t) + \dots + P_3(t) = C = 0$.

We calculate the characteristic polynomial of matrix A :

$$\begin{aligned} \det(\delta I_4 - \mathbf{A}) &= \delta^4 + a_1 \delta^3 + a_2 \delta^2 + a_3 \delta + a_4 \\ &= \delta^4 + 3(\lambda + \mu_0) \delta^3 + (3\lambda^2 + 3\mu_0^2 + 4\lambda\mu_0) \delta^2 + (\lambda + \mu_0)(\lambda^2 + \mu_0^2) \delta. \end{aligned} \quad (5.3)$$

Let $\lambda = 0$. Then from (5.3) it follows that $\det(\delta I_4 - \mathbf{A}) = \delta \cdot (\delta + \mu_0)^3$. The last equality means that for a sufficiently small $\lambda \geq 0$, all roots of the polynomial (5.3) will be real. Condition (b2) of Theorem 5.1 excludes this situation.

Now let's use the reasoning from paragraph (a) of Theorem 5.1. By assumption, at point $t = 0$ we have

$$P_3(0) = -(P_0(0) + \dots + P_2(0)).$$

Then the vector $\mathbf{P}_{10} = (P_{10}(0), P_{11}(0), P_{12}(0))^T$ can always be chosen so that $P_3(0) = -(P_{10}(0) + P_{11}(0) + P_{12}(0)) > 0$. (To do this, you need to multiply the vector \mathbf{W}_0 by a suitable nonzero factor.) In this case, for $\epsilon = 0$, $\lambda + \mu_0 = 0$, and the vector of initial conditions $(P_{10}(0), P_{11}(0), P_{12}(0), P_3(0))^T$, system (2.1) has a periodic solution.

By assumption $\epsilon \neq 0$. It is now possible to apply the well-known Poincare Theorem, according to which, if the linear part (it does not depend on ϵ) of a nonlinear system has a periodic solution $\mathbf{v}(t)$, then for a sufficiently small $|\epsilon| > 0$ the nonlinear system will also have a periodic solution $\mathbf{w}(t) \neq \mathbf{v}(t)$ (in our case, this is system (2.1)).

(c) Finally we will prove the existence of the limit cycle. Now assume that $\lambda + \mu_0 < 0$. According to paragraph (b), for sufficiently small $|\lambda + \mu_0| > 0$ and $|\epsilon| > 0$, system (2.1) has only unstable equilibrium points and its solution is periodic. Therefore, there must exist a compact positively invariant set $\mathbb{H} \subset \mathbb{R}^4$ with respect to system (2.1) [2, 6].

By \mathbb{H} denote the ball of radius R with center at point $(r_0, \dots, r_3)^T$.

Let

$$V(\mathbf{P}) = (P_0 - r_0)^2 + \dots + (P_3 - r_3)^2 - R^2$$

be a real continuously differentiable function such that $\dot{V}(\mathbf{P}) \leq 0$ in \mathbb{H} . Here $r_0, \dots, r_3, R \in \mathbb{R}$; $\mathbf{P} = (P_0, \dots, P_3)^T \in \mathbb{R}^4$. Then we have

$$\dot{V}(\mathbf{P}) = \epsilon \mu_0 P_3 \mathbf{P}^T (\mathbf{B}\mathbf{Q} + \mathbf{Q}^T \mathbf{B}^T) \mathbf{P} + \mathbf{P}^T (\mathbf{A} + \mathbf{A}^T) \mathbf{P} + F(r_0, \dots, r_3, \epsilon \mu_0, \mathbf{P}),$$

where $F(r_0, \dots, r_3, \epsilon \mu_0, \mathbf{P})$ is a polynomial of degree 2. Let us choose constants r_0, \dots, r_3 so that function $\mathbf{P}^T (\mathbf{A}^T + \mathbf{A}) \mathbf{P} + F(r_0, \dots, r_3, \epsilon \mu_0, \mathbf{P})$ is negative definite. Then there exists the radius R such that $\dot{V}(\mathbf{P}) \leq 0$ in \mathbb{H} .

Let \mathbb{E} be the set of all points in \mathbb{H} such that $\dot{V}(\mathbb{E}) = 0$. Let \mathbb{M} be the largest invariant set in \mathbb{E} . Then according to LaSalle's Theorem [2, 6] every solution starting in \mathbb{H} approaches \mathbb{M} as $t \rightarrow +\infty$.

Due to the choice of function $V(\mathbf{P})$ and restriction (b2), the boundary defined by the condition $\dot{V}(\mathbb{E}) = 0$ should not contain equilibrium points. Since all such points are points of the type of unstable focus, then any trajectory starting from the region \mathbb{H} must approach not the equilibrium point, but to the limit cycle. \square

Of course, in this case, the coordinates of solution $\mathbf{P}(t) = (P_0(t), P_1(t), P_2(t), P_3(t))^T \in \mathbb{R}^4$ no longer correspond to the practical meaning that was declared in the introduction. However, there may be other applications of system (2.1) in which restrictions $P_0(t) \geq 0, \dots, P_3(t) \geq 0$ will be removed.

Fig.5.1 and Fig.5.2 were constructed for the case when condition (2.2) was valid. If condition $P_0(0) + \dots + P_3(0) > 1$ is satisfied, then several limit cycles are formed (see Fig.5.3 and Fig.5.4). Note that in case (2.2), starting from different initial conditions (for fixed λ, μ_0, ϵ) leads the trajectory of system (2.1) to one limit cycle. At the same time, if $P_0(0) + \dots + P_3(0) > 1$, then starting from different initial conditions (for fixed λ, μ_0, ϵ) leads the trajectories of system (2.1) to different limit cycles.

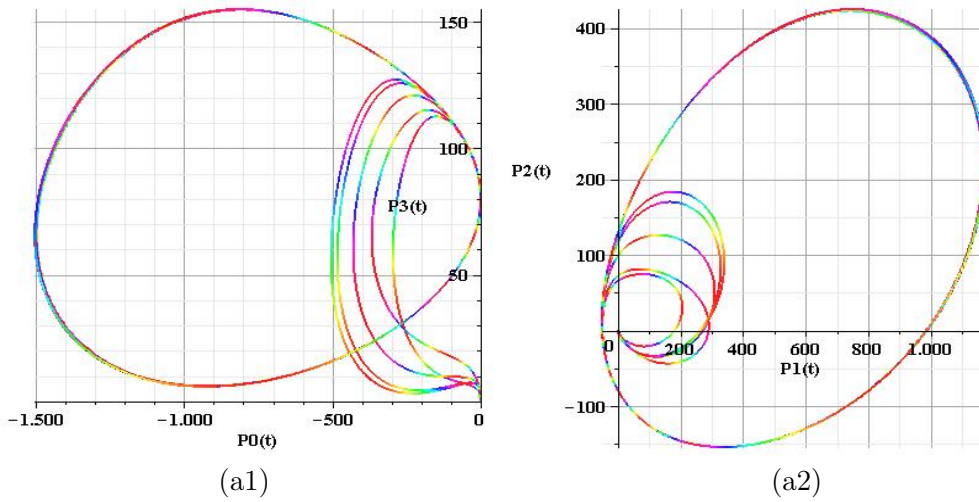


Fig. 5.1. Birth of a limit cycle in system (2.1) with parameters $\lambda = 0.2, \mu_0 = -0.8, \epsilon = -0.01$, and different initial data $P_0(0) + \dots + P_3(0) = 1$. Projections of the cycle onto: (a1) the plane $P_0 - P_3$ and (a2) the plane $P_1 - P_2$.

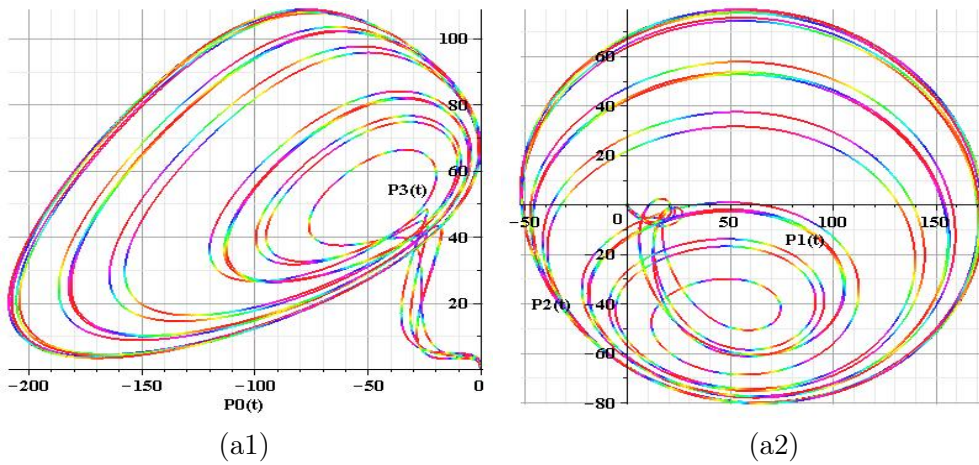


Fig. 5.2. Birth of a limit cycle in system (2.1) with parameters $\lambda = 0.2, \mu_0 = -0.4, \epsilon = -0.01$, and different initial data such that $P_0(0) + \dots + P_3(0) = 1$. Projections of the cycle onto: (a1) the plane $P_0 - P_3$ and (a2) the plane $P_1 - P_2$.

At first sight it appears that Fig.5.1 and Fig.5.2 and Fig.5.3 are significantly different from each other. In fact, the only difference is that for constructing trajectories in Fig.5.1 and Fig.5.2 five starting points were used. While for the trajectories in Fig.5.3 twenty starting points were used.

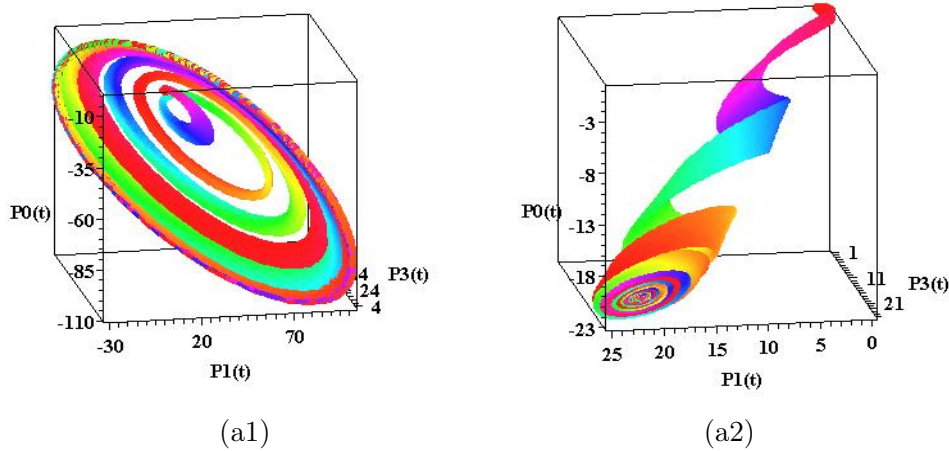


Fig. 5.3. Birth of limit cycles in system (2.1) under fixed initial conditions $P_1(0) = [0.1, 0, 15, \dots, 0.95, 1.0]$ with parameters: (a1) $\lambda = 3, \mu_0 = -5, \epsilon = -0.01$, (a2) $\lambda = 4, \mu_0 = -5, \epsilon = -0.01$, and the same initial data $P_0(0), P_2(0), P_3(0)$.

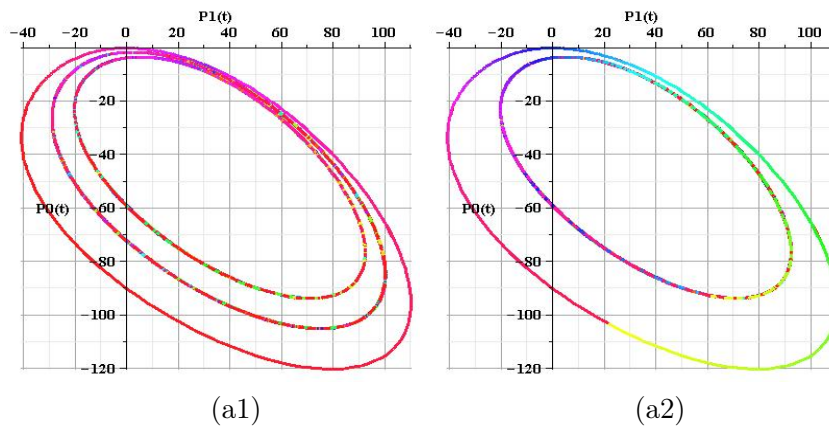


Fig. 5.4. Confirmation of the existence of limit cycles in system (2.1). Three different starting points lead to three cycles (a1) and three other starting points lead to two cycles (a2). (The middle cycle on the chart (a1) is attracted to the inner cycle on the chart (a2).)

6. Conclusion

Model (2.1) showed that, under the conditions of Theorem 2.1, variables P_0, \dots, P_3 reach their equilibrium values over time. (Mathematically, this means that the equilibrium point of system (2.1) is asymptotically stable [6, 9].) The only question is how much this result corresponds to the real dynamics of sales?

Note that our task was a mathematical study of solutions to systems (2.1) (with restrictions (2.2) or restrictions (4.1), (4.2)) and systems (3.1). Therefore, we do not in any way pretend to derive important economic conclusions. We only

hope that, perhaps, some of our results will be useful to economists who specialize in accelerating the dynamics of market trade.

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