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Odd and Even Functions in the Design Problem of New Chaotic Attractors

Vasiliy Ye. Belozyorov
*Department of Applied Mathematics,
 Dnipro National University, Gagarin's avenue, 72,
 49010, Dnipro, Ukraine
 belozvye2017@gmail.com*

Svetlana A. Volkova
*Department of Computer Information Technologies,
 Ukrainian State University of Science and Technologies,
 Academician Lazaryan's Street, 2, 49010, Dnipro, Ukraine
 svolkovav2017@gmail.com*

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Let $\mathbb{A} \subset \mathbb{R}^n$ be a chaotic attractor generated by a quadratic system of ordinary differential equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. A method for constructing new chaotic attractors based on the attractor \mathbb{A} is proposed. The idea of the method is to replace the state vector $\mathbf{x} = (x_1, \dots, x_n)^T$ located on the right side of the original system with new vector $\mathbf{u}(\mathbf{x})$; where $\mathbf{u}(\mathbf{x}) = K \cdot (h_1(x_1), \dots, h_n(x_n))^T$, $K \in \mathbb{R}^{n \times n}$, and $h_i(x_i)$ are odd power functions; $i = 1, \dots, n$. (In other words, a state feedback $\mathbf{x} \rightarrow \mathbf{u}(\mathbf{x})$ is introduced into the right side of the system under study: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \rightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{u}(\mathbf{x}))$.) As a result, the newly obtained system generates new chaotic attractors, which are topologically not equivalent (generally speaking) to the attractor \mathbb{A} . In addition, for an antisymmetric neural ODE system with a homoclinic orbit connected at a saddle point, the conditions for the occurrence of chaotic dynamics are found.

Keywords: System of ordinary autonomous differential equations; power function; state feedback; limit cycle; homoclinic orbit; chaotic attractor.

1. Introduction

The traditional methods of studying chaotic systems are such that for a given dynamical system, using various definitions, it is possible to judge whether this system is chaotic or not. Since various engineering applications require more and more use of chaotic processes, it is necessary to create new methods that will allow expanding the range of chaotic systems used in modeling compared to previously known systems. (See, for example, [Banks, 1999; Belozyorov, 2011a, 2011b, 2014, 2015a, 2015b, 2016, 2017, 2018, 2019; Hu, 2009; Kai *et al.*, 2017; Liu *et al.*, 2007; Liu *et al.*, 2010; Lu & Chen, 2006;

Wang *et al.*, 2008; Wang *et al.*, 2009; Wang *et al.*, 2018; Wang & Chen, 2013; Yang & Chen, 2008; Yang *et al.*, 2010; Yu *et al.*, 2010; Yu & Hu, 2010; Zhou *et al.*, 2017; Zhu & Du, 2019].)

To this list we can add the article [Lai *et al.*, 2018], which investigates a new chaotic 3D system with three nonlinearities, as well as the paper [Lai, 2020], which considers a 4D chaotic system with various coexisting attractors and special nonlinearities with multiple zeros. Particularly noteworthy is the article [Lai *et al.*, 2020b], which investigates a new chaotic system without equilibria and an infinite number of coexisting hidden attractors for

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various initial conditions. A new modification of Chua's system with an absolute element was studied in [Wu *et al.*, 2021].

It should be emphasized that among all methods of chaotic dynamics especially important are the methods which allow designing models adequately describing the chaotic processes. Now we will point out several well-known scientific directions in which the use of chaotic dynamics tools has led to good practical results.

One such direction is the creation of chaotic memristor systems. For example, in article [Gu *et al.*, 2021], a memristor system without equilibrium positions was considered; in [Liu *et al.*, 2019], a memristor was used to increase the complexity of the system, and in [Lai *et al.*, 2020a], a new memristor system was constructed from a Lorenz-type system.

Noteworthy is the work [Wei *et al.*, 2017], which investigates the hidden hyperchaos in an electronic circuit in a 5D self-exciting homopolar disc dynamo, review [Pham & Hong, 2020] of dynamic models for axially moving systems, as well as the article [Benkoudir *et al.*, 2021] in the field of security communication, which considers a 3D chaotic system with five nonlinearities. The controlled weather model and its physical applications have been considered in [Sooraksa & Chen, 2018].

In recent years, methods related to the use of neural ODEs have become widespread for modeling dynamic processes [Goodfellow *et al.*, 2017; Chen *et al.*, 2019; Chang *et al.*, 2019; Li, 2020; Belozyorov & Dantsev, 2020, 2022; Belozyorov *et al.*, 2021]. We especially note the article [Haber & Ruthotto, 2019], in which the authors proposed new forward propagation techniques inspired by systems of ODEs that overcome this challenge and lead to well-posed learning problems for arbitrarily deep neural networks.

Note that the type of nonlinearities (activation functions) involved in the construction of neural ODEs and their relationships with each other play a key role in studying the problem of approximating trajectories described by known time series: these time series are modeled by solving these ODEs.

Any chaotic process generates a chaotic attractor that determines the dynamics of this process. Most modern models of chaotic processes in the world around us are described by quadratic differential equations. It is clear that such modeling does not always adequately describe the behavior of the

process. Therefore, a problem arises: based on the known attractor of the quadratic model, it is necessary to build a new dynamic model that will generate the attractor of the real process. The system of neural ODEs usually serves as such a model.

Finally, to conclude this brief introduction, the book [Wang *et al.*, 2021] should be mentioned.

In this book, written by a team of well-known authors, special three-dimensional autonomous quadratic chaotic systems are studied without equilibrium, with only stable equilibria, with equilibrium curves, with equilibrium surfaces, and even with any number and different types of equilibria. The importance of the results obtained in this book lies in the following. Applying methods of the present article to the systems investigated in the book [Wang *et al.*, 2021] it is possible to sharply increase the number of new chaotic attractors with highly nontrivial properties.

It should be said that the quality of modeling depends significantly on the activation functions used. There are many studies devoted to the influence of activation functions on the quality of approximation of continuous functions. In this work, we will use power activation functions. (The motivation for this choice will be explained below.)

Introduce the power functions [Belozyorov & Dantsev, 2020]:

$$g_o(u, \alpha \vee \beta) = \begin{cases} \text{if } (u < 0 \text{ and } \beta > 0) \text{ then } -(-u)^\beta, \\ \text{if } (u \geq 0 \text{ and } \alpha > 0) \text{ then } u^\alpha \end{cases} \quad (1)$$

and

$$g_e(u, \alpha \vee \beta) = \begin{cases} \text{if } (u < 0 \text{ and } \beta > 0) \text{ then } (-u)^\beta, \\ \text{if } (u \geq 0 \text{ and } \alpha > 0) \text{ then } u^\alpha. \end{cases} \quad (2)$$

Definition 1 [Belozyorov & Dantsev, 2020]. Representation (1) (2) is called an odd (even) activation function.

Notice. If the parity or oddness of the activation function is not important, then the subscript "o" (or "e") can be omitted.

Consider the following linear combination of activation functions:

$$\mathcal{H}(u, \mathbf{s}) = g(u, \alpha_0 \vee \beta_0) + s_1 g(u, \alpha_1 \vee \beta_1) + \dots + s_k g(u, \alpha_k \vee \beta_k) \in \mathbb{R}[u], \quad (3)$$

where $\mathbb{R}[u]$ is the set of real power functions of one variable u ; $\mathbf{s} = (s_1, \dots, s_k, \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k)$; $s_i \in \mathbb{R}$; $\alpha_0 > \alpha_i > 0, \beta_0 > \beta_i > 0$; $i = 1, \dots, k$.

Definition 2. Representation (3) is called the mixed power activation function.

2. Attraction Regions for Quadratic ODEs

Let us recall some results obtained in [Belozyorov *et al.*, 2021].

Let $A = (a_{ij}), B_1, \dots, B_n \in \mathbb{R}^{n \times n}$ be real matrices and c_1, \dots, c_n be real numbers. In addition, let the matrices $B_1 = (b_{ij}^{(1)}), \dots, B_n = (b_{ij}^{(n)})$ be symmetrical; $i, j = 1, \dots, n$.

$\mathbf{x} = (x_1, \dots, x_n)^T$ denotes an arbitrary vector of \mathbb{R}^n . Consider the system of ordinary autonomous quadratic differential equations

$$\left\{ \begin{array}{l} \dot{x}_1(t) = c_1 + \sum_{j=1}^n a_{1j}x_j(t) + \mathbf{x}^T(t)B_1\mathbf{x}(t) \\ \vdots \\ \dot{x}_n(t) = c_n + \sum_{j=1}^n a_{nj}x_j(t) + \mathbf{x}^T(t)B_n\mathbf{x}(t) \end{array} \right. \quad (4)$$

of order n with the vector of initial values $\mathbf{x}^T(0) = (x_{10}, \dots, x_{n0})$.

Assume that the region of attraction for the solutions of system (4) is a ball

$$\mathbb{B} \equiv (x_1 + \gamma_1)^2 + \dots + (x_n + \gamma_n)^2 - R^2 \leq 0$$

of radius R with center at point $(-\gamma_1, \dots, -\gamma_n)^T$.

Let also the elements of matrices B_1, \dots, B_n satisfy the following three groups of restrictions:

C_n^1 one-term restrictions

$$b_{ii}^{(i)} x_i^3 \equiv 0; \quad i = 1, \dots, n; \quad (5)$$

$2C_n^2$ two-term restrictions

$$b_{jj}^{(i)} x_i x_j^2 + b_{ij}^{(j)} x_i x_j^2 \equiv 0; \quad i \neq j; \quad i, j = 1, \dots, n; \quad (6)$$

C_n^3 three-term restrictions

$$\begin{aligned} & b_{jk}^{(i)} x_i x_j x_k + b_{ik}^{(j)} x_i x_j x_k + b_{ij}^{(k)} x_i x_j x_k \\ & \equiv 0; \quad i \neq j \neq k; \quad i, j, k = 1, \dots, n. \end{aligned} \quad (7)$$

As shown in [Belozyorov *et al.*, 2021], for small values of n system (4), taking into account restrictions (5)–(7), we get the following form:

$n = 3$

$$\left\{ \begin{array}{l} \dot{x}(t) = c_1 + a_{11}x + \dots + a_{13}z + b_{12}xy + b_{13}xz + b_{22}y^2 + b_{23}yz + b_{33}z^2, \\ \dot{y}(t) = c_2 + a_{21}x + \dots + a_{23}z - b_{12}x^2 - b_{22}xy + c_{13}xz + c_{23}yz + c_{33}z^2, \\ \dot{z}(t) = c_3 + a_{31}x + \dots + a_{33}z - b_{13}x^2 - (b_{23} + c_{13})xy - b_{33}xz - c_{23}y^2 - c_{33}yz; \end{array} \right. \quad (8)$$

$n = 4$

$$\left\{ \begin{array}{l} \dot{x}(t) = c_1 + a_{11}x + \dots + a_{14}u + b_{12}xy + b_{13}xz + b_{14}xu + b_{22}y^2 \\ \quad + b_{23}yz + b_{24}yu + b_{33}z^2 + b_{34}zu + b_{44}u^2, \\ \dot{y}(t) = c_2 + a_{21}x + \dots + a_{24}u - b_{12}x^2 - b_{22}xy + c_{13}xz + c_{14}xu \\ \quad + c_{23}yz + c_{24}yu + c_{33}z^2 + c_{34}zu + c_{44}u^2, \\ \dot{z}(t) = c_3 + a_{31}x + \dots + a_{34}u - b_{13}x^2 - (b_{23} + c_{13})xy - b_{33}xz \\ \quad + d_{14}xu - c_{23}y^2 - c_{33}yz + d_{24}yu + d_{34}zu + d_{44}u^2, \\ \dot{u}(t) = c_4 + a_{41}x + \dots + a_{44}u - b_{14}x^2 - (b_{24} + c_{14})xy - (b_{34} + d_{14})xz \\ \quad - b_{44}xu - c_{24}y^2 - (c_{34} + d_{24})yz - c_{44}yu - d_{34}z^2 - d_{44}zu. \end{array} \right. \quad (9)$$

Note that Eqs. (8) and (9) are presented in this detailed form solely for the convenience of users. In the case of arbitrary n , the system that satisfies the conditions (5)–(7) looks like this:

$$\dot{\mathbf{x}}(t) = \mathbf{c} + (A + B(\mathbf{x}) - B^T(\mathbf{x})) \cdot \mathbf{x}.$$

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Here

$$B(\mathbf{x}) = \begin{pmatrix} 0 & b_{12}^1 x_1 + b_{22}^1 x_2 & b_{13}^1 x_1 + b_{23}^1 x_2 + b_{33}^1 x_3 & \cdots & \sum_{i=1}^n b_{in}^1 x_i \\ 0 & 0 & b_{13}^2 x_1 + b_{23}^2 x_2 + b_{33}^2 x_3 & \cdots & \sum_{i=1}^n b_{in}^2 x_i \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & \sum_{i=1}^n b_{in}^{n-1} x_i \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$b_{ij}^k \in \mathbb{R}; i, j, k \in \{1, \dots, n\}$. (It is clear that $\mathbf{x}^T \cdot (B(\mathbf{x}) - B^T(\mathbf{x})) \cdot \mathbf{x} \equiv 0$.)

The method for finding the radius R of sphere \mathbb{B} and its center $(-\gamma_1, \dots, -\gamma_n)^T$ is presented in [Belozyorov *et al.*, 2021].

For example, let $n = 3$. Then, for the existence of a compact region of attraction, it is sufficient that the following symmetric matrix

$$H(\gamma_1, \gamma_2, \gamma_3) = \begin{pmatrix} a_{11} & \frac{a_{12} + a_{21}}{2} & \frac{a_{13} + a_{31}}{2} \\ \frac{a_{12} + a_{21}}{2} & a_{22} & \frac{a_{23} + a_{32}}{2} \\ \frac{a_{13} + a_{31}}{2} & \frac{a_{23} + a_{32}}{2} & a_{33} \end{pmatrix} + \gamma_1 \begin{pmatrix} 0 & \frac{b_{12}}{2} & \frac{b_{13}}{2} \\ \frac{b_{12}}{2} & b_{22} & \frac{b_{23}}{2} \\ \frac{b_{13}}{2} & \frac{b_{23}}{2} & b_{33} \end{pmatrix} \\ + \gamma_2 \begin{pmatrix} -b_{12} & -\frac{b_{22}}{2} & \frac{c_{13}}{2} \\ -\frac{b_{22}}{2} & 0 & \frac{c_{23}}{2} \\ \frac{c_{13}}{2} & \frac{c_{23}}{2} & c_{33} \end{pmatrix} + \gamma_3 \begin{pmatrix} -b_{13} & -\frac{b_{23} + c_{13}}{2} & -\frac{b_{33}}{2} \\ -\frac{b_{23} + c_{13}}{2} & -c_{23} & -\frac{c_{33}}{2} \\ -\frac{b_{33}}{2} & -\frac{c_{33}}{2} & 0 \end{pmatrix}$$

be negative definite.

The last restriction can be achieved if for some real $\gamma_1, \gamma_2, \gamma_3$ the conditions of Sylvester (see [Khalil, 1996])

$$\left\{ \begin{array}{l} \Delta_1(\gamma_1, \gamma_2, \gamma_3) = a_{11} - b_{12}\gamma_2 - b_{13}\gamma_3 < 0, \\ \Delta_2(\gamma_1, \gamma_2, \gamma_3) = (a_{11} - b_{12}\gamma_2 - b_{13}\gamma_3)(a_{22} + b_{22}\gamma_1 - c_{23}\gamma_3) \\ \quad - \frac{(a_{12} + a_{21} + b_{12}\gamma_1 - b_{22}\gamma_2 - (b_{23} + c_{13})\gamma_3)^2}{4} > 0, \\ \Delta_3(\gamma_1, \gamma_2, \gamma_3) = \det H(\gamma_1, \gamma_2, \gamma_3) < 0 \end{array} \right. \quad (10)$$

are true.

3. Attraction Regions for Power ODEs

We introduce the following designations:

$$\begin{aligned} h_1(u_1) &= \mathcal{H}(u_1, \mathbf{s}_1) = g_o(u_1, \alpha_{10} \vee \beta_{10}) + s_{11}g(u_1, \alpha_{11} \vee \beta_{11}) + \cdots + s_{1k}g(u_1, \alpha_{1k} \vee \beta_{1k}), \\ h_2(u_2) &= \mathcal{H}(u_2, \mathbf{s}_2) = g_o(u_2, \alpha_{20} \vee \beta_{20}) + s_{21}g(u_2, \alpha_{21} \vee \beta_{21}) + \cdots + s_{2k}g(u_2, \alpha_{2k} \vee \beta_{2k}), \\ &\vdots \\ h_n(u_n) &= \mathcal{H}(u_n, \mathbf{s}_n) = g_o(u_n, \alpha_{n0} \vee \beta_{n0}) + s_{n1}g(u_n, \alpha_{n1} \vee \beta_{n1}) + \cdots + s_{nk}g(u_n, \alpha_{nk} \vee \beta_{nk}) \end{aligned}$$

and

$$\mathbf{H}(\mathbf{u}) = (h_1(u_1), \dots, h_n(u_n))^T.$$

Here $h(u_i)$; $i = 1, \dots, n$, are mixed activation functions (3).

Now, instead of system (4), we will consider the following system

$$\begin{cases} \dot{x}_1(t) = c_1 + \sum_{j=1}^n a_{1j}h_j(x_j) + \mathbf{H}^T(\mathbf{x})B_1\mathbf{H}(\mathbf{x}) \\ \vdots \\ \dot{x}_n(t) = c_n + \sum_{j=1}^n a_{nj}h_j(x_j) + \mathbf{H}^T(\mathbf{x})B_n\mathbf{H}(\mathbf{x}). \end{cases} \quad (11)$$

In a more general form, system (11) can be represented as the following state feedback system:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{c} + A \cdot \mathbf{u}(\mathbf{x}) + [\mathbf{u}(\mathbf{x})^T B_1 \mathbf{u}(\mathbf{x}), \dots, \\ &\quad \mathbf{u}(\mathbf{x})^T B_n \mathbf{u}(\mathbf{x})]^T, \end{aligned} \quad (12)$$

where $\mathbf{u}(\mathbf{x}) = K \cdot \mathbf{H}(\mathbf{x})$, $K \in \mathbb{R}^{n \times n}$ is a feedback matrix.

Below we will use the following well-known result:

Theorem 1 (LaSalle's Theorem [Khalil, 1996]). *Let $\mathbb{H} \subset \mathbb{R}^n$ be a compact set that is positively invariant with respect to (11). Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(\mathbf{x}) \leq 0$ (or $\dot{V}(\mathbf{x}) \geq 0$) in \mathbb{H} . Let \mathbb{E} be the set of all points in \mathbb{H} where $\dot{V}(\mathbf{x}) = 0$. Let \mathbb{M} be the largest invariant set in \mathbb{E} . Then every solution starting in \mathbb{H} approaches \mathbb{M} as $t \rightarrow +\infty$.*

We will assume that $K = I$. Let s_1, \dots, s_n be unknown real constants. Let us construct from matrices A and B_1, \dots, B_n of system (11) the following matrix:

$$F(s_1, \dots, s_n) := \frac{(A^T + A)}{2} + s_1 B_1 + \cdots + s_n B_n.$$

Introduce also the following function $V : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$\begin{aligned} V(x_1, \dots, x_n) &= \frac{g_e(x_1, (\alpha_{10} \vee \beta_{10}) + 1)}{(\alpha_{10} \vee \beta_{10}) + 1} + s_{11} \frac{g(x_1, (\alpha_{11} \vee \beta_{11}) + 1)}{(\alpha_{11} \vee \beta_{11}) + 1} \\ &+ s_{12} \frac{g(x_1, (\alpha_{12} \vee \beta_{12}) + 1)}{(\alpha_{12} \vee \beta_{12}) + 1} + \cdots + s_{1k} \frac{g(x_1, (\alpha_{1k} \vee \beta_{1k}) + 1)}{(\alpha_{1k} \vee \beta_{1k}) + 1} + s_1 x_1 \\ &+ \frac{g_e(x_2, (\alpha_{20} \vee \beta_{20}) + 1)}{(\alpha_{20} \vee \beta_{20}) + 1} + s_{21} \frac{g(x_2, (\alpha_{21} \vee \beta_{21}) + 1)}{(\alpha_{21} \vee \beta_{21}) + 1} \\ &+ s_{22} \frac{g(x_2, (\alpha_{22} \vee \beta_{22}) + 1)}{(\alpha_{22} \vee \beta_{22}) + 1} + \cdots + s_{2k} \frac{g(x_2, (\alpha_{2k} \vee \beta_{2k}) + 1)}{(\alpha_{2k} \vee \beta_{2k}) + 1} + s_2 x_2 \\ &+ \cdots + \frac{g_e(x_n, (\alpha_{n0} \vee \beta_{n0}) + 1)}{(\alpha_{n0} \vee \beta_{n0}) + 1} + s_{n1} \frac{g(x_n, (\alpha_{n1} \vee \beta_{n1}) + 1)}{(\alpha_{n1} \vee \beta_{n1}) + 1} \\ &+ s_{n2} \frac{g(x_n, (\alpha_{n2} \vee \beta_{n2}) + 1)}{(\alpha_{n2} \vee \beta_{n2}) + 1} + \cdots + s_{nk} \frac{g(x_n, (\alpha_{nk} \vee \beta_{nk}) + 1)}{(\alpha_{nk} \vee \beta_{nk}) + 1} + s_n x_n. \end{aligned}$$

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(We remind the reader that the product $g_o(u) \cdot u$ of the odd function $g_o(u)$ and the odd function u is the even function $f_e(u) := g_o(u) \cdot u$ [Belozyorov & Dantsev, 2020].)

Thus, due to the fact that $(\alpha_{ik} \vee \beta_{ik}) + 1 > 1$, the derivatives $\partial V(x_1, \dots, x_n) / \partial x_i$; $i = 1, \dots, n$, exist and are finite. Therefore, it is obvious that

$$\begin{aligned} \dot{V}_t &= (h_1(x_1) + s_1) \cdot \dot{x}_1(t) + (h_2(x_2) + s_2) \cdot \dot{x}_2(t) \\ &+ \dots + (h_n(x_n) + s_n) \cdot \dot{x}_n(t). \end{aligned} \quad (13)$$

(Here the derivative \dot{V}_t is defined by virtue of Eqs. (11).)

Let R be a positive constant. We define the set $\mathbb{B}_R \subset \mathbb{R}^n$ as follows:

$$\begin{aligned} \mathbb{B}_R &:= \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \\ &V(x_1, \dots, x_n) - R^2 \leq 0\}. \end{aligned} \quad (14)$$

Introduce the following sets:

$$\mathbb{D}_- := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \dot{V}_t(x_1, \dots, x_n) \leq 0\}, \quad (15)$$

$$\mathbb{D}_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \dot{V}_t(x_1, \dots, x_n) \geq 0\} \quad (16)$$

and

$$\mathbb{L} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \dot{V}_t(x_1, \dots, x_n) = 0\}. \quad (17)$$

Theorem 2. *Suppose that for system (11) the following conditions:*

- (1) $c_1 = \dots = c_n = 0$;
- (2) *the matrices B_i satisfy restrictions (5)–(7), in which substitutions $x_i \rightarrow h_i(x_i)$ were made, $i = 1, \dots, n$;*
- (3) *there are real constants s_1^*, \dots, s_n^* such that the matrix $F(s_1^*, \dots, s_n^*)$ is negative definite, are satisfied.*

Then there exists the compact region of attraction $\mathbb{H} = \mathbb{D}_+ \neq \emptyset$ for trajectories of system (11).

Proof. Condition (1) guarantees that system (11) has an equilibrium point $(0, \dots, 0)^T \in \mathbb{R}^n$. (Theorem 1 is valid only for systems with equilibria.)

Further, condition (2) guarantees that the forms $\mathbf{H}^T(\mathbf{x})B_i\mathbf{H}(\mathbf{x})$ have the type of quadratic forms $\mathbf{x}^T B_i \mathbf{x}$ presented in (8) and (9); $i = 1, \dots, n$.

In addition, the same condition allows us to say that the function

$$\begin{aligned} \dot{V}_t(x_1, \dots, x_n) &= (s_1, \dots, s_n) \cdot \mathbf{A}\mathbf{H} \\ &+ \mathbf{H}^T F(s_1, \dots, s_n) \mathbf{H} \end{aligned} \quad (18)$$

[see (13)] contains only linear and quadratic terms and does not contain cubic terms.

It remains only to clarify condition (3). So, let there exist numbers s_1^*, \dots, s_n^* such that the matrix $F(s_1^*, \dots, s_n^*)$ is negative definite.

Proof of condition (3) is split into two parts.

(3a) The function $V(x_1, \dots, x_n)$ for $s_i = 0$, $s_{jk} = 0$ is positive definite and $\lim_{\|\mathbf{x}\| \rightarrow \infty} V(x_1, \dots, x_n) = \infty$. Thus, by virtue of the construction of the function $V(x_1, \dots, x_n)$, the sets \mathbb{B}_R and \mathbb{L} are compact. Therefore, we can choose R such that $\mathbb{B}_R \cap \mathbb{D}_- \neq \emptyset$ and $\mathbb{L} \subset \mathbb{B}_R$. Then, in the region $\mathbb{B}_R \cap \mathbb{D}_-$, we can assert that $V(x_1(t), \dots, x_n(t))$ is a decreasing function of t . Since $V(x_1(t), \dots, x_n(t))$ is continuous on the compact set \mathbb{B}_R , it is bounded from below on \mathbb{B}_R . Therefore, $V(x_1(t), \dots, x_n(t))$ has a finite limit as $t \rightarrow \infty$. Then, according to Theorem 1, we can assume that $\mathbb{H} = \mathbb{B}_R \cap \mathbb{D}_-$ and \mathbb{H} is the compact region of attraction for trajectories of system (11).

(3b) Now we choose the radius R so large that the set $\mathbb{B}_R \cap \mathbb{D}_+ = \mathbb{D}_+ \neq \emptyset$. (Note that, by virtue of (3), the set \mathbb{D}_+ is compact. Therefore, we have $\mathbb{L} \subset \mathbb{B}_R$.) Then, in the domain $\mathbb{B}_R \cap \mathbb{D}_+$ the function $V(x_1(t), \dots, x_n(t))$ is an increasing function of t . Since the function $V(x_1(t), \dots, x_n(t))$ is continuous on the compact set \mathbb{D}_+ , it is bounded from above on \mathbb{D}_+ and has a finite limit as $t \rightarrow \infty$.

Thus, from items (3a) and (3b) it follows that, regardless of the starting point $\mathbf{x}^T(0) \in \mathbb{R}^n$, the trajectory $V(x_1(t), \dots, x_n(t))$ will be attracted to the boundary $\dot{V}_t(x_1, \dots, x_n) = 0$ [this is \mathbb{L} ; see (18)] of the compact set \mathbb{D}_+ . This means that there exists an attractor belonging to the region \mathbb{D}_+ . (An equilibrium point can act as such an attractor.) ■

Comment 1. If in Theorem 2 condition (2) is not satisfied, then instead of variables x_i , variables $y_i = \mu_i x_i$, $\mu_i \neq 0$ should be introduced into system (4); $i = 1, \dots, n$. After that, we can return to system (11), but with respect to variables y_1, \dots, y_n .

Comment 2. If condition (3) of Theorem 2 is not satisfied, then one should take $K \neq I$ and instead of the matrix $F(s_1, \dots, s_n)$, it is necessary to work

with the matrix

$$F(K, s_1, \dots, s_n) := \frac{(K^T A^T + AK)}{2} + K^T (s_1 B_1 + \dots + s_n B_n) K$$

and system (12).

3.1. Attractors generated by system (11)

In this subsection, we assume that in the substitutions $x \rightarrow h_1(x) = x^{\alpha_1 \vee \beta_1}$, $y \rightarrow h_2(y) = y^{\alpha_2 \vee \beta_2}$, $z \rightarrow h_3(z) = z^{\alpha_3 \vee \beta_3}$ the functions $h_i(\dots)$ are odd activation functions; $i = 1, \dots, 3$.

(1) Consider the classical Lorenz system

$$\begin{cases} \dot{x}(t) = -\sigma x + \sigma y, \\ \dot{y}(t) = rx - y - xz, \\ \dot{z}(t) = -bz + xy, \end{cases} \quad (19)$$

where $\sigma = 10$, $r = 24.75$, $b = -8/3$.

Let us check the conditions of Theorem 2 for this system. We have

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix},$$

$$B_1 = 0, \quad B_2 = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Assume that $s_1 = 0$, $s_2 = 0$, and $s_3 = -(\sigma + r)$. Then, we have

$$\begin{aligned} F &= \frac{(A + A^T)}{2} + s_1 B_1 + s_2 B_2 + s_3 B_3 \\ &= \begin{pmatrix} -\sigma & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}. \end{aligned}$$

It is obvious that the matrix F is negative definite. Thus, for system (19) all conditions of Theorem 2 are satisfied, and as the region of attraction, we can take the ball $\mathbb{B}_R = x^2 + y^2 + (z - (\sigma + r))^2 \leq R^2$ [see (14)] of large enough radius R such that $\mathbb{D}_+ \subset \mathbb{B}_R$. Various modifications of the classical Lorenz attractor are shown in Fig. 1.

(2) Consider the following system [Belozyorov, 2014]:

$$\begin{cases} \dot{x}(t) = -2x + 7y^2 + 13z^2, \\ \dot{y}(t) = 1.81x + 7y + 10z - 3xy, \\ \dot{z}(t) = -10y + 7z - 3xz. \end{cases} \quad (20)$$

For system (20), we have

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 1.81 & 7 & 10 \\ 0 & -10 & 7 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & -\frac{3}{2} & 0 \\ -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 \end{pmatrix}.$$

From here it follows that

$$\begin{aligned} F &= \frac{(A + A^T)}{2} + s_1 B_1 + s_2 B_2 + s_3 B_3 \\ &= \begin{pmatrix} -2 & \frac{1.81 - 3s_2}{2} & -\frac{3s_3}{2} \\ \frac{1.81 - 3s_2}{2} & 7 + s_1 & 0 \\ -\frac{3s_3}{2} & 0 & 7 + 13s_1 \end{pmatrix}. \end{aligned}$$

Thus, in order for the matrix F to be negative definite, it suffices to put $s_1 = -8$, $s_2 = 40$, $s_3 = 0.6$. In this case all conditions of Theorem 2 are satisfied. Various modifications of the attractor of system (20) are shown in Fig. 2.

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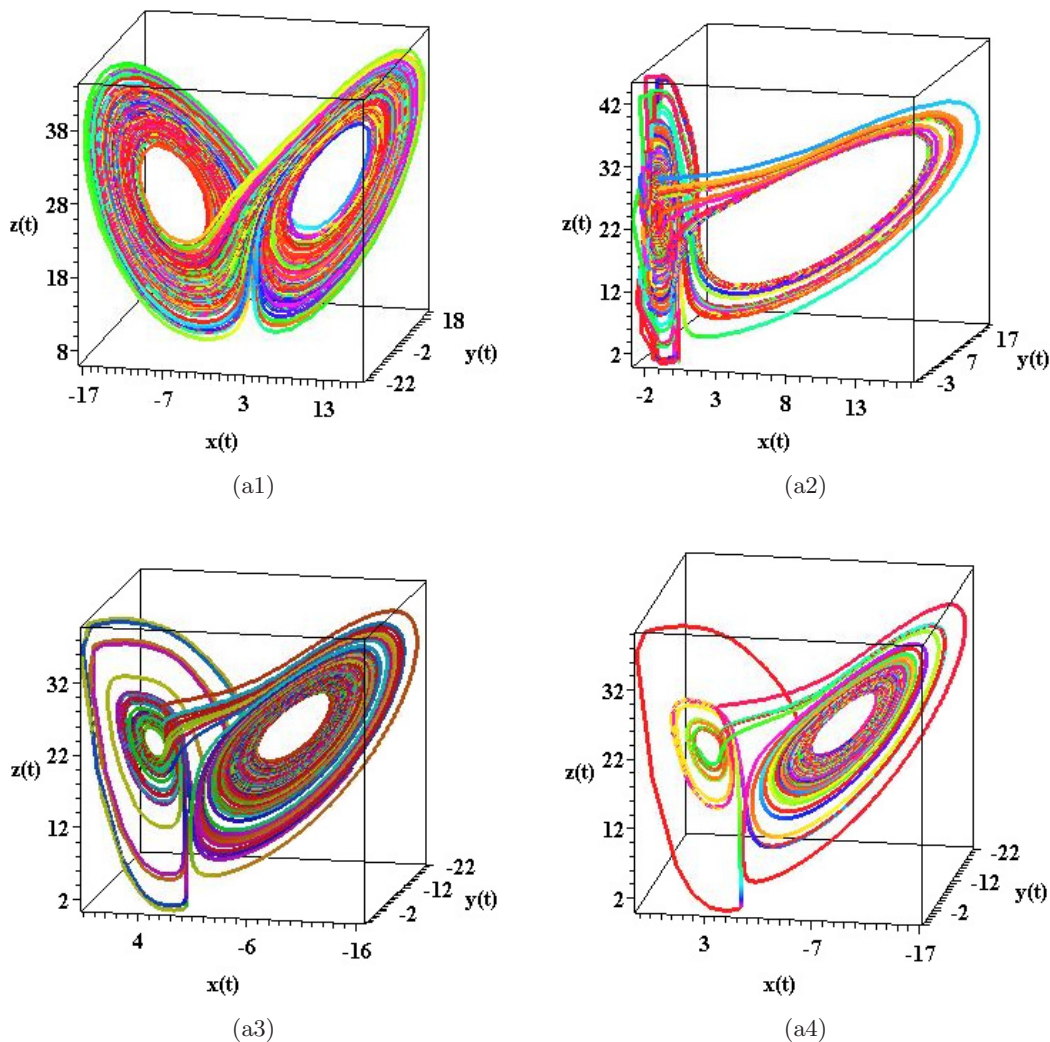


Fig. 1. The attractor of systems (19) (a1) and its modifications: (a2) $x \rightarrow x^{5\sqrt{1}}$, $y \rightarrow y^{5\sqrt{1}}$, $z \rightarrow z^{1\sqrt{1}}$; (a3) $x \rightarrow x^{2\sqrt{1}}$, $y \rightarrow y^{5\sqrt{1}}$, $z \rightarrow z^{1\sqrt{1}}$ and (a4) $x \rightarrow x^{2\sqrt{1}}$, $y \rightarrow y^{5\sqrt{1}}$, $z \rightarrow z^{1.005\sqrt{1}}$.

(3) Consider the following Chen system [Wang & Chen, 2013]:

$$\begin{cases} \dot{x}(t) = 11x - yz, \\ \dot{y}(t) = -15y + xz, \\ \dot{z}(t) = x - z + xy. \end{cases} \quad (21)$$

System (21) does not satisfy condition (2) of Theorem 2. Therefore, we make a change of variables $x \rightarrow x/\sqrt{2}$, $y \rightarrow y$, $z \rightarrow z$. In this case, we have

$$\begin{cases} \dot{x}(t) = 11x - \sqrt{2}yz, \\ \dot{y}(t) = -15y + \frac{xz}{\sqrt{2}}, \\ \dot{z}(t) = x - z + \frac{xy}{\sqrt{2}}. \end{cases} \quad (22)$$

System (22) already satisfies condition (2) of Theorem 2. [To simplify the notation in the newly obtained system (22), we left the same variables as in system (21).]

We introduce a positive definite function

$$V(x, y, z) = x^2 + y^2 + z^2.$$

Then, taking system (22) into account, we have

$$\dot{V}_t = 22x^2 - 30y^2 + 2xz - 2z^2.$$

Surface $\dot{V}_t = 0$ is a cone with the vertex at the origin, and the region \mathbb{D}_- will be the interior of this cone. Thus, according to Theorem 2 $\forall R > 0$, we have

$$\mathbb{B}_R \cap \mathbb{D}_- \neq \emptyset.$$

Various modifications of the Chen attractor are shown in Fig. 3.

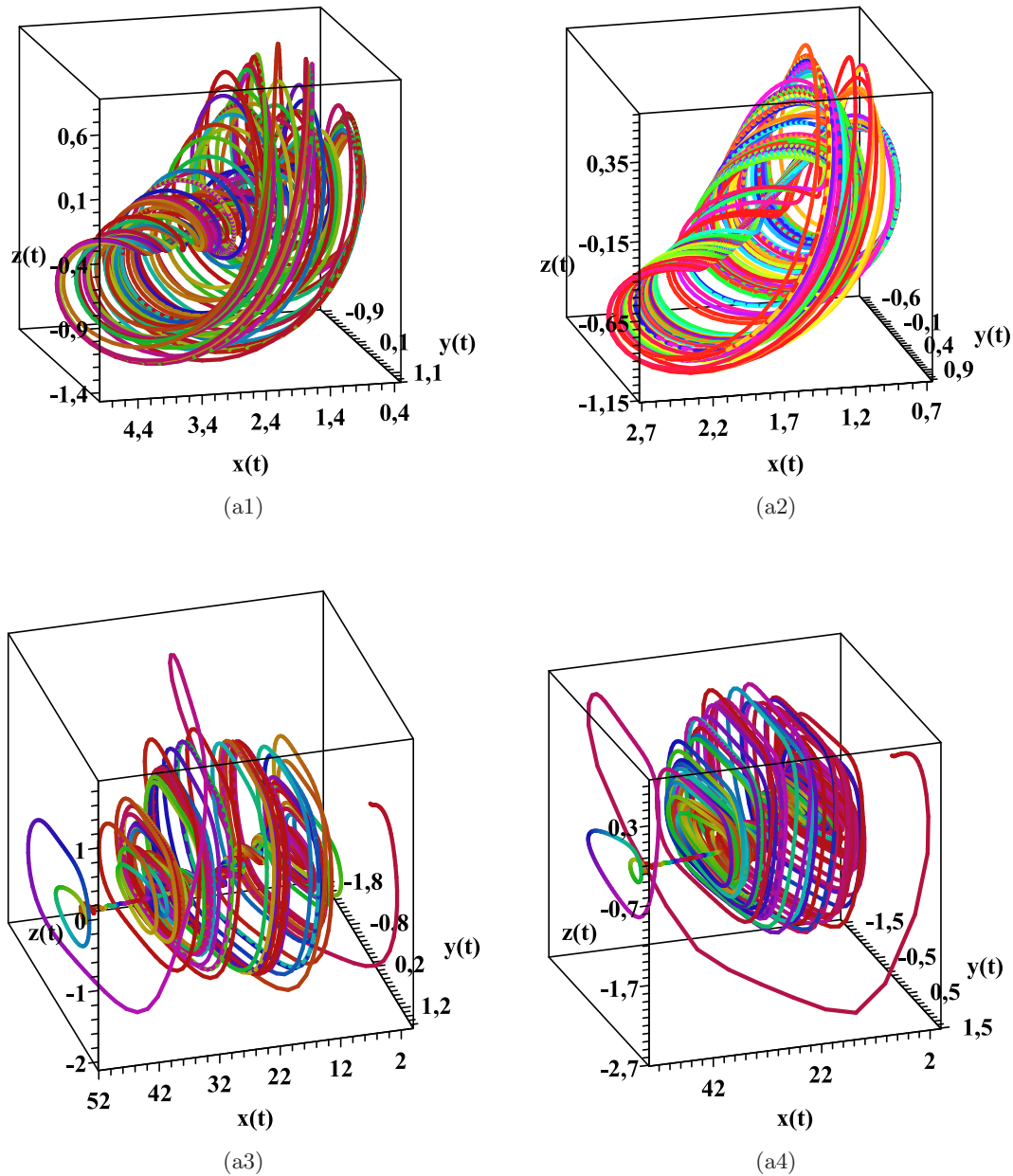


Fig. 2. The attractor of system (20) (a1) and its modifications: (a2) $x \rightarrow x^{1.71\sqrt{1}}$, $y \rightarrow y^{1\sqrt{1}}$, $z \rightarrow z^{1.33\sqrt{1}}$; (a3) $x \rightarrow x^{0.27\sqrt{1}}$, $y \rightarrow y^{3\sqrt{3}}$, $z \rightarrow z^{1.62\sqrt{1}}$ and (a4) $x \rightarrow x^{0.27\sqrt{1}}$, $y \rightarrow y^{5\sqrt{5}}$, $z \rightarrow z^{1.62\sqrt{1}}$.

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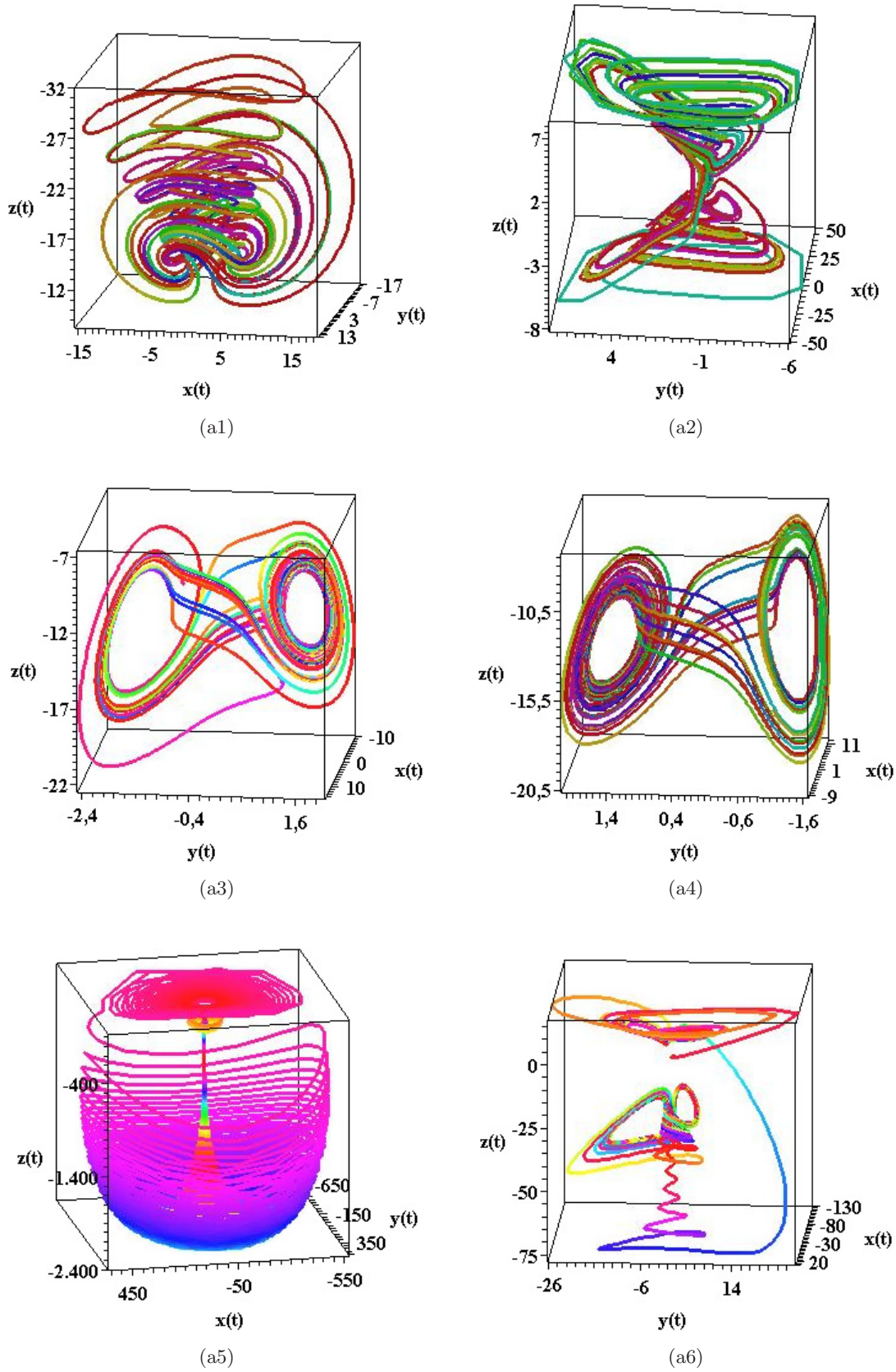


Fig. 3. The attractor of systems (21) (a1) and its modifications: (a2) $x \rightarrow x^{1\vee 1}$, $y \rightarrow y^{3\vee 3}$, $z \rightarrow z^{3\vee 3}$; (a3) $x \rightarrow x^{1\vee 1}$, $y \rightarrow y^{3\vee 3}$, $z \rightarrow z^{1\vee 1}$; (a4) $x \rightarrow x^{1\vee 1}$, $y \rightarrow y^{3\vee 5}$, $z \rightarrow z^{1\vee 1}$; (a5) $x \rightarrow x^{1\vee 1}$, $y \rightarrow y^{1\vee 0.95}$, $z \rightarrow z^{1.5\vee 0.85}$ and (a6) $x \rightarrow x^{0.95\vee 0.55}$, $y \rightarrow y^{1\vee 0.95}$, $z \rightarrow z^{1.5\vee 0.85}$.

Odd and Even Functions in the Design Problem of New Chaotic Attractors

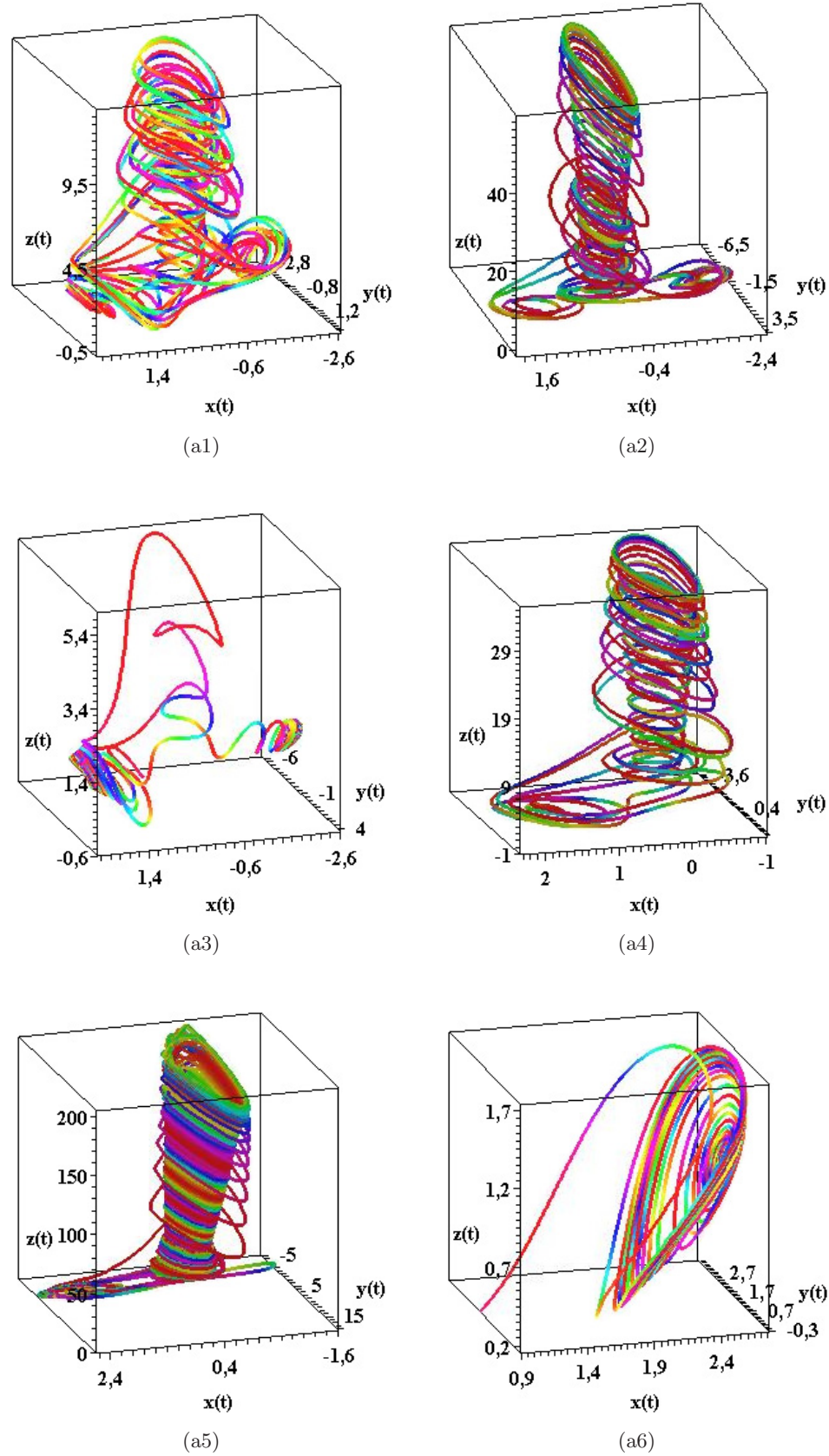


Fig. 4. The attractor of systems (23) (a1) and its modifications: (a2) $x \rightarrow x^{1\vee 1}$, $y \rightarrow y^{1.5\vee 1.2}$, $z \rightarrow z^{1.5\vee 1.2}$; (a3) $x \rightarrow x^{1\vee 1}$, $y \rightarrow y^{1\vee 1}$, $z \rightarrow z^{3\vee 3}$; (a4) $x \rightarrow x^{1\vee 1}$, $y \rightarrow y^{1.3\vee 1.3}$, $z \rightarrow z^{4.3\vee 1.3}$; (a5) $x \rightarrow x^{1\vee 1}$, $y \rightarrow y^{0.75\vee 2.5}$, $z \rightarrow z^{1.5\vee 0.72}$ and (a6) $x \rightarrow x^{1\vee 1}$, $y \rightarrow y^{0.275\vee 2.2}$, $z \rightarrow z^{1.5\vee 0.2}$.

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(4) Consider another Chen system [Wang & Chen, 2013]

$$\begin{cases} \dot{x}(t) = -0.4x + y, \\ \dot{y}(t) = x + 0.3y - xz, \\ \dot{z}(t) = -0.1z + y^2 - 1. \end{cases} \quad (23)$$

System (23) does not satisfy the conditions of Theorem 2. However, applying the methods

$$\begin{cases} \dot{x}_1(t) = \alpha_{11} \cdot f_1 \left(\sum_{j=1}^n \beta_{1j}^{(1)} x_j + \gamma_1^{(1)} \right) + \dots + \alpha_{1,n+1} \cdot f_1 \left(\sum_{j=1}^n \beta_{n+1,j}^{(1)} x_j + \gamma_{n+1}^{(1)} \right) \\ \vdots \\ \dot{x}_n(t) = \alpha_{n1} \cdot f_n \left(\sum_{j=1}^n \beta_{1j}^{(n)} x_j + \gamma_1^{(n)} \right) + \dots + \alpha_{n,n+1} \cdot f_n \left(\sum_{j=1}^n \beta_{n+1,j}^{(n)} x_j + \gamma_{n+1}^{(n)} \right). \end{cases} \quad (24)$$

with the known vector of initial values $(x_{10}, \dots, x_{n0})^T$. (Here $\alpha_{ik}, \beta_{kj}^{(1)}, \dots, \beta_{kj}^{(n)}, \gamma_k^{(1)}, \dots, \gamma_k^{(n)}$ are known real constants; $i = 1, \dots, n$; $k = 1, \dots, n+1$.)

System (24) will be called the system of neural ODEs. (The right parts of system (24) are constructed according to well-known results on the approximation of continuous functions of several variables (see, for example, [Belozyorov & Dantsev, 2022]).)

Let us introduce two real functions $\psi(u)$ and $\phi(u)$ of one variable u having a common root a : $\psi(a) = \phi(a) = 0$. Let $\phi(u)$ be a continuous increasing function on the interval (a, ∞) . (It is clear that $\psi(u) > 0$ for $u > a$.) Similarly, let $-\psi(-u)$ be a continuous increasing function on the interval $(-\infty, a)$. (It is clear that $-\psi(-u) < 0$ for $u < a$.)

We also define functions $\theta(u)$ and $\eta(u)$ in exactly the same way as functions $\phi(u)$ and $\psi(u)$.

In what follows, for simplicity, we will assume that $a = 0$. Moreover, in order to generalize the results obtained in [Belozyorov & Dantsev, 2022], we restrict the rate of growth of the function $\phi(u)$ ($\psi(u)$) by the following conditions:

$$\begin{aligned} \lim_{u \rightarrow \infty} (\phi(u) - u^\alpha) < 0, \\ \alpha \geq 0 \left(\lim_{u \rightarrow -\infty} (\psi(-u) - (-u)^\beta) < 0, \beta \geq 0 \right). \end{aligned} \quad (25)$$

described above have made it possible to obtain new types of attractors from the attractor of system (23) (see Fig. 4).

4. On the Existence of Homoclinic Orbits in Neural ODEs

Let real functions $f_i(u)$; $i = 1, \dots, n$, of one variable u be defined and be continuous on the set $(-\infty, \infty)$.

Consider the following system of autonomous differential equations

Introduce the following activation functions [compare with formulas (1), (2)]:

$$\begin{aligned} h_i(u) &:= \phi_i(u) \vee \psi_i(u) \\ &= \begin{cases} \text{if } (u < 0) \text{ then } -\psi_i(-u), \\ \text{if } (u \geq 0) \text{ then } \phi_i(u); \end{cases} \quad i = 1, \dots, n, \end{aligned} \quad (26)$$

or

$$\begin{aligned} g_i(u) &:= \theta_i(u) \vee \eta_i(u) \\ &= \begin{cases} \text{if } (u < 0) \text{ then } \eta_i(-u), \\ \text{if } (u \geq 0) \text{ then } \theta_i(u); \end{cases} \quad i = 1, \dots, n. \end{aligned} \quad (27)$$

As in cases (1) [or (2)], we will call representation (26) [or (27)] an odd (or even) activation function. (For example, the activation function ReLU [Goodfellow *et al.*, 2017], widely known in the theory of neural networks, is an odd function (26) for which $\phi_i(u) = \alpha_i u$, $\alpha_i > 0$, $u \geq 0$; $\psi_i(u) = \beta_i u = -\beta_i \cdot (-u)$, $u < 0$, $\beta_i > 0$, and $\alpha_i \neq \beta_i$; $i = 1, \dots, n$.)

We introduce the following vector functions:

$$\begin{aligned} \mathbf{h}(\mathbf{x}) &:= (h_1(x_1), \dots, h_n(x_n))^T \\ &= (\phi_1(x_1) \vee \psi_1(x_1), \dots, \phi_n(x_n) \vee \psi_n(x_n))^T \end{aligned} \quad (28)$$

and

$$\begin{aligned} \mathbf{g}(\mathbf{x}) &:= (g_1(x_1), \dots, g_n(x_n))^T \\ &= (\theta_1(x_1) \vee \eta_1(x_1), \dots, \theta_n(x_n) \vee \eta_n(x_n))^T. \end{aligned} \quad (29)$$

Definition 3. Vector function $\mathbf{f}(\mathbf{x}) = (f_1(x_1), \dots, f_n(x_n))^T$ is called the odd (even) activation function

if all its components $f_i(x_i)$ are odd (even); $i = 1, \dots, n$.

Suppose that in system (24) $f_1(u), \dots, f_n(u)$ is the odd activation functions. Then, as shown in [Belozyorov & Dantsev, 2022], using the recalculation of parameters, system (24) can be represented in the following form

$$\begin{cases} \dot{x}_1(t) = f_1\left(\sum_{j=1}^n \beta_{1j}^{(1)} x_j + \gamma_1^{(1)}\right) + \dots + f_1\left(\sum_{j=1}^n \beta_{n+1,j}^{(1)} x_j + \gamma_{n+1}^{(1)}\right) \\ \vdots \\ \dot{x}_n(t) = f_n\left(\sum_{j=1}^n \beta_{1j}^{(n)} x_j + \gamma_1^{(n)}\right) + \dots + f_n\left(\sum_{j=1}^n \beta_{n+1,j}^{(n)} x_j + \gamma_{n+1}^{(n)}\right). \end{cases} \quad (30)$$

(To simplify the notation, we left in system (30) the previous designations of the parameters of system (24).)

Consider the simplest version of system (30)

$$\begin{cases} \dot{x}_1(t) = f_1(a_{11}x_1 + \dots + a_{1n}x_n + p_1), \\ \dot{x}_2(t) = f_2(a_{21}x_1 + \dots + a_{2n}x_n + p_2), \\ \vdots \\ \dot{x}_n(t) = f_n(a_{n1}x_1 + \dots + a_{nn}x_n + p_n), \end{cases} \quad (31)$$

where a_{ij}, p_i are known real constants; $i, j = 1, \dots, n$.

Along with system (31), we will also consider the system

$$\begin{cases} \dot{x}_1(t) = a_{11}f_1(x_1) + \dots + a_{1n}f_n(x_n), \\ \dot{x}_2(t) = a_{21}f_1(x_1) + \dots + a_{2n}f_n(x_n), \\ \vdots \\ \dot{x}_n(t) = a_{n1}f_1(x_1) + \dots + a_{nn}f_n(x_n). \end{cases} \quad (32)$$

(Note that if $\det A \neq 0$, then with the help of the change of variables $\mathbf{x} \rightarrow A\mathbf{x} + \mathbf{p}$ it is always possible to pass from system (31) to system (32) and vice versa; $\mathbf{p} \in \mathbb{R}^n$. Therefore, in what follows, we restrict ourselves to the study of system (32).)

We introduce the notation

$$\mathbf{f}(\mathbf{x}) = (f_1(x_1), \dots, f_n(x_n))^T,$$

$$\mathbf{h}(\mathbf{x}) = (h_1(x_1), \dots, h_n(x_n))^T,$$

$$\mathbf{g}(\mathbf{x}) = (g_1(x_1), \dots, g_n(x_n))^T$$

and

$$\mathbf{f}(\mathbf{x}) = \mathbf{h}(\mathbf{x}) + K\mathbf{g}(\mathbf{x}), \quad (33)$$

where the functions $h_i(x_i)$ and $g_i(x_i)$; $i = 1, \dots, n$, are defined by formulas (26) or (27). The matrix

$$K = \text{diag}[(-1)^{k_1}, \dots, (-1)^{k_n}],$$

where $k_i = 0$ or $k_i = 1$, and situations such that $\forall i \in \{1, \dots, n\} k_i = 0$ or $\forall i \in \{1, \dots, n\} k_i = 1$ are excluded.

Taking into account (33), we introduce the following function

$$\begin{aligned} V(x_1, \dots, x_n) &= \int f_1(s_1) ds_1 + \dots + \int f_n(s_n) ds_n \\ &= V_1(x_1, \dots, x_n) + V_2(x_1, \dots, x_n), \end{aligned}$$

where

$$V_1(x_1, \dots, x_n) = \int h_1(s_1) ds_1 + \dots + \int h_n(s_n) ds_n,$$

$$V_2(x_1, \dots, x_n) = \int (-1)^{k_1} g_1(s_1) ds_1$$

$$+ \dots + \int (-1)^{k_n} g_n(s_n) ds_n.$$

(34)

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The integral of the function $\mathbf{h}(\mathbf{x})$ is defined by formulas:

(1) if $h_i(s_i)$ is even then

$$\begin{aligned} \text{if } x_i \geq 0 \text{ then } \int h_i(s_i) ds_i &= \int_0^{x_i} \phi_i(s_i) ds_i \\ \text{else } \int h_i(s_i) ds_i &= \int_{x_i}^0 \psi_i(s_i) ds_i; \end{aligned}$$

(2) if $h_i(s_i)$ is odd then

$$\begin{aligned} \text{if } x_i \geq 0 \text{ then } \int h_i(s_i) ds_i &= \int_0^{x_i} \phi_i(s_i) ds_i \\ \text{else } \int h_i(s_i) ds_i &= - \int_{x_i}^0 \psi_i(s_i) ds_i. \end{aligned}$$

Similarly for the function $\mathbf{g}(\mathbf{x})$, we use formulas:

(1) if $g_i(s_i)$ is even then

$$\begin{aligned} \text{if } x_i \geq 0 \text{ then } \int g_i(s_i) ds_i &= \int_0^{x_i} \theta_i(s_i) ds_i \\ \text{else } \int g_i(s_i) ds_i &= \int_{x_i}^0 \eta_i(s_i) ds_i; \end{aligned}$$

(2) if $g_i(s_i)$ is odd then

$$\begin{aligned} \text{if } x_i \geq 0 \text{ then } \int g_i(s_i) ds_i &= \int_0^{x_i} \theta_i(s_i) ds_i \\ \text{else } \int g_i(s_i) ds_i &= - \int_{x_i}^0 \eta_i(s_i) ds_i. \end{aligned}$$

Here functions $\theta_i(s_i)$ and $\eta_i(s_i)$ have the same properties as functions $\phi_i(s_i)$ and $\psi_i(s_i)$; $i = 1, \dots, n$.

We also introduce the following restrictions:

$$\begin{aligned} \lim_{x_i \rightarrow \infty} (\phi_i(x_i) - \theta_i(x_i)) &> 0, \\ \lim_{x_i \rightarrow -\infty} (\psi_i(x_i) - \eta_i(x_i)) &> 0; \quad i = 1, \dots, n. \end{aligned} \quad (35)$$

(Sometimes we will use the notation $\mathbf{h}(\mathbf{x}) > \mathbf{g}(\mathbf{x})$ instead of inequalities (35).)

Theorem 3. *Assume that in system (32) the matrix $A = \{a_{ij}\}$ is antisymmetric: $A^T + A = 0$. Let all components of the vector function $\mathbf{h}(\mathbf{x})$ be odd activation functions. We also assume that for vector functions $\mathbf{h}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ conditions (35) are satisfied. Then any solution $\mathbf{x}(t, \mathbf{x}_0)$ of system (32) is periodic. In addition, if all components of the vector function $\mathbf{g}(\mathbf{x})$ is odd, then there exists a*

vector of initial conditions \mathbf{x}_0^ such that trajectory $\mathbf{x}(t, \mathbf{x}_0^*)$ is the homoclinic orbit connected at equilibrium point $\mathbf{0}$.*

Proof. It is known [Belozyorov & Dantsev, 2020] that the integral of the odd activation function $h_i(x_i)$ is the even activation function. Therefore, we have $V_1(x_1, \dots, x_n) \geq 0$.

Compute the total derivative with respect to t of the function $V(x_1, \dots, x_n)$:

$$\begin{aligned} \dot{V}(x_1, \dots, x_n) &= 0.5 \left[\dot{x}_1(t) \frac{\partial V(x_1, \dots, x_n)}{\partial x_1} \right. \\ &\quad \left. + \dots + \dot{x}_n(t) \frac{\partial V(x_1, \dots, x_n)}{\partial x_n} \right] \\ &\quad + 0.5 \left[\frac{\partial V(x_1, \dots, x_n)}{\partial x_1} \dot{x}_1(t) \right. \\ &\quad \left. + \dots + \frac{\partial V(x_1, \dots, x_n)}{\partial x_n} \dot{x}_n(t) \right] \\ &= 0.5 \mathbf{f}^T(\mathbf{x})(A^T + A)\mathbf{f}(\mathbf{x}) \\ &= 0. \end{aligned} \quad (36)$$

Further, by virtue of inequalities (35) and the fact that $V_1(x_1, \dots, x_n) \geq 0$, we have

$$\begin{aligned} \lim_{x_i \rightarrow \infty} \int_{-x_i}^{x_i} f_i(s_i) ds_i \\ = \lim_{x_i \rightarrow \infty} \int_{-x_i}^0 f_i(s_i) ds_i + \lim_{x_i \rightarrow \infty} \int_0^{x_i} f_i(s_i) ds_i \\ \geq 0; \quad i = 1, \dots, n. \end{aligned}$$

This means that $\lim_{x_i \rightarrow \infty} V(x_1, \dots, x_n) \geq 0$; $i = 1, \dots, n$. In addition, from (36) it follows that for a sufficiently large value $\|\mathbf{x}_0\|$, we have $V(x_1(t), \dots, x_n(t)) = \text{const} = V(x_{10}, \dots, x_{n0}) = V(\mathbf{x}_0) > 0$. This implies that the set $\mathbb{S} = \{V(x_1(t), \dots, x_n(t)) - V(x_{10}, \dots, x_{n0}) = 0\}$ is compact. Therefore, any trajectory $\mathbf{x}(t, \mathbf{x}_0)$ of system (32) is closed and periodic.

Now we will assume that all components of the vector function $\mathbf{g}(\mathbf{x})$ are odd.

Let the vector of initial conditions $\mathbf{x}_0^* \neq \mathbf{0}$ be such that $V(\mathbf{x}_0^*) = 0$. Therefore, if the function $\mathbf{g}(\mathbf{x})$ is odd, then the function $V(\mathbf{x})$ is even. From here it follows that $V(\mathbf{x}_0^*) = V(-\mathbf{x}_0^*) = 0$ and in addition, $V(\mathbf{0}) = 0$. Consequently, there is a trajectory of the system (32), which leaves some point

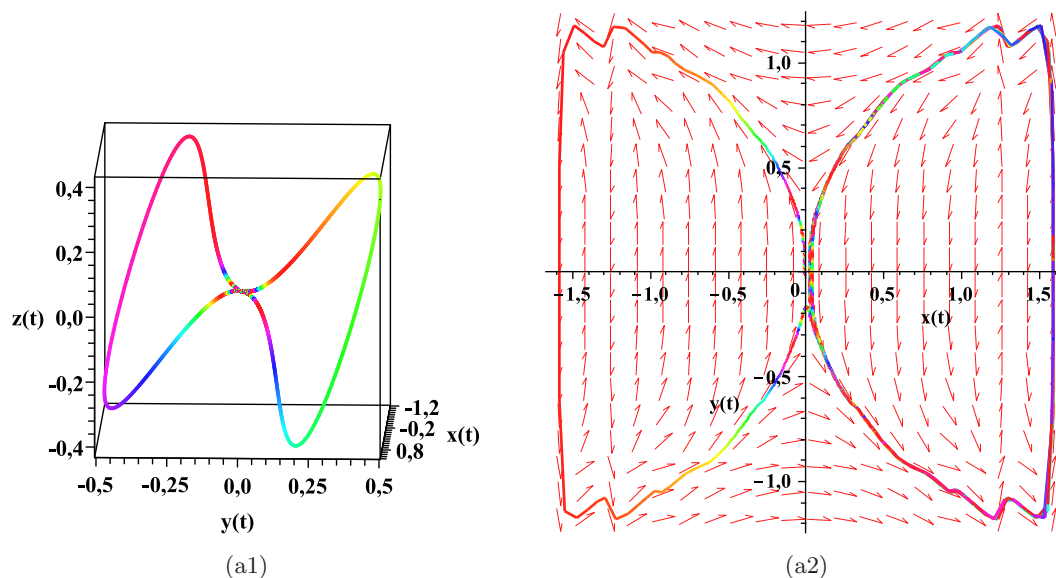


Fig. 5. Homoclinic orbits in systems (37)(a1) and (38)(a2).

$\mathbf{x}_{c0} \neq \mathbf{0}$ arbitrarily close to $\mathbf{x}_0^* \approx \mathbf{0}$ and as $t \rightarrow \pm\infty$ approaches the point $\mathbf{0}$.

Let us introduce instead of function $V(x_1(t), \dots, x_n(t))$ the function $V(x_1(-\tau), \dots, x_n(-\tau))$. As a result, instead of analyzing the equation $V(\mathbf{x}(t)) = V(\mathbf{x}_{c0}) > 0$, we come to an analysis of the equation $V(\mathbf{x}(-\tau)) = V(\mathbf{x}_{c0}) > 0$. Thus, based on the structure of even function $V(\mathbf{x}(t))$, we have $V(\mathbf{x}(\tau)) = V(\mathbf{x}(-t)) = V(\mathbf{x}(t)) = V(\mathbf{x}_{c0}) > 0$ and $\lim_{x_i \rightarrow \infty} V(\mathbf{x}_{c0}) \rightarrow \lim_{x_i \rightarrow \infty} V(\mathbf{x}_0^*) = 0$. This means the existence of a homoclinic orbit. ■

4.1. Examples of homoclinic orbits

Consider the following system [see (32), (33)]:

$$\begin{cases} \dot{x}(t) = (y^5 + 3z^5) + (-y^3 + 3z), \\ \dot{y}(t) = (-x^5 + z^5) + (x^3 + z), \\ \dot{z}(t) = (-3x^5 - y^5) + (3x^3 + y^3). \end{cases} \quad (37)$$

Here

$$A = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -3 & -1 & 0 \end{pmatrix},$$

$$K = \text{diag}(-1, -1, 1),$$

$$\mathbf{h}(\mathbf{x}) = (x^5, y^5, z^5)^T,$$

$$\mathbf{g}(\mathbf{x}) = (x^3, y^3, z^1)^T.$$

Consider also another system with odd non-power nonlinearities:

$$\begin{cases} \dot{x}(t) = -\frac{y^5}{0.5 + 14 \sin^2(10y)} - y^3, \\ \dot{y}(t) = \frac{x^5}{0.5 + 14 \sin^2(10x)} - x. \end{cases} \quad (38)$$

Here

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$K = \text{diag}(-1, 1),$$

$$\mathbf{h}(\mathbf{x}) = \left(\frac{x^5}{0.5 + 14 \sin^2(10x)}, \frac{y^5}{0.5 + 14 \sin^2(10y)} \right)^T,$$

$$\mathbf{g}(\mathbf{x}) = (x, y^3)^T.$$

Graphs of homoclinic trajectories of these systems are presented in Fig. 5.

Thus, for both systems (37) and (38) the conditions of Theorem 3 are satisfied.

5. On Conditions for the Existence of Chaos in Neural ODEs

It is known that the following question often arises when modeling chaotic processes: can the created model generate chaotic behavior? We will try to

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answer this question for the following system:

$$\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}) + (A - D_0)\mathbf{h}(\mathbf{x}). \quad (39)$$

Here matrices $A, D_0 \in \mathbb{R}^{n \times n}$, the matrix A is anti-symmetric, and the matrix $D_0 = \text{diag}(d_1 \geq 0, \dots, d_n \geq 0)$; the vector functions $\mathbf{g}(\mathbf{x}) = (g_1(x_1), \dots, g_n(x_n))^T$, $\mathbf{h}(\mathbf{x}) = (h_1(x_1), \dots, h_n(x_n))^T \in \mathbb{R}^n$ are defined by formulas (29) and (28). (Immediately, we note that in Eq. (39), function (28) should only be odd, and function (29) can be either even or odd or mixed.)

Further, in this section, we will assume that

$$\det \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\mathbf{0}} \neq 0. \quad (40)$$

Let us construct for system (39) a positive definite function $V_1(x_1, \dots, x_n)$ [see (34)]. Then, we will have $\dot{V}_{1t}(x_1, \dots, x_n) = 0.5 \cdot (\mathbf{h}^T(\mathbf{x})\mathbf{g}(\mathbf{x}) + (\mathbf{g}^T(\mathbf{x})\mathbf{h}(\mathbf{x}) - \mathbf{h}^T(\mathbf{x})D_0\mathbf{h}(\mathbf{x})).$

We denote by \mathbb{W} the set of all points from \mathbb{R}^n such that $\dot{V}_{1t}(x_1, \dots, x_n) \leq 0$. Let also $\mathbb{L} \subset \mathbb{W}$ be the set of all points in \mathbb{W} such that $\dot{V}_{1t}(x_1, \dots, x_n) = 0$. We also denote by $\mathbb{X} \subset \mathbb{W}$ an open set in \mathbb{W} such that $\forall \mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{X} \dot{V}_{1t}(x_1, \dots, x_n) < 0$. (Thus, $\mathbb{X} \cup \mathbb{L} = \mathbb{W}$.)

Let $r = \min(d_1, \dots, d_n) \geq 0$.

Theorem 4. *Assume that the components of the vector functions $\mathbf{h}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ satisfy conditions (35) ($\mathbf{h}(\mathbf{x}) > \mathbf{g}(\mathbf{x}) \geq \mathbf{x}$), and the components of the function $\mathbf{h}(\mathbf{x})$ are only odd. Let also the point $\mathbf{0}$ be a unique equilibrium point for system (39) in \mathbb{W} for which condition (40) is satisfied.*

Suppose that:

- (1) point $\mathbf{0}$ is a saddle point;
- (2) for the components of vector $\mathbf{h}(\mathbf{x})$, there are constants $\alpha_i > 0$ and $\beta_i > 0$ such that conditions (25) are satisfied, $i = 1, \dots, n$;
- (3) there exists a value of parameter $r = r_c \geq 0$ such that for an arbitrary vector of initial data $(x_{10}, \dots, x_{n0})^T \in \mathbb{X}$ the solutions $x_1(x_{10}, \dots, x_{n0}, t), \dots, x_n(x_{10}, \dots, x_{n0}, t)$ of system (39) satisfy equality

$$\liminf_{t \rightarrow \infty} V_1(x_1(x_{10}, \dots, x_{n0}, t), \dots, x_n(x_{10}, \dots, x_{n0}, t)) = 0. \quad (41)$$

Then in system (39) there exist chaotic dynamics.

Proof. We use the well-known Cauchy–Bunyakovsky inequality: for any continuous functions $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x}) \in \mathbb{R}^n$, we have

$$\mathbf{u}^T(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \leq \|\mathbf{u}(\mathbf{x})\| \cdot \|\mathbf{v}(\mathbf{x})\|.$$

Let us apply this inequality to the function $\dot{V}_{1t}(x_1, \dots, x_n)$. Then, taking into account inequalities (35), we obtain the estimate

$$\begin{aligned} \dot{V}_{1t}(x_1, \dots, x_n) &\leq \|\mathbf{h}(\mathbf{x})\| \cdot \|\mathbf{g}(\mathbf{x})\| - r\|\mathbf{h}(\mathbf{x})\|^2 \\ &\leq (1 - r)\|\mathbf{h}(\mathbf{x})\|^2. \end{aligned}$$

Thus, if the number $r = r_0$ satisfies condition $1 < r$, then $\dot{V}_{1t}(x_1, \dots, x_n) < 0$ and the solutions of system (39) are bounded. Therefore, for $r = r_0$ the set \mathbb{W} is a compact positively invariant set with respect to system (39).

According to LaSalla's theorem every solution of system (39) starting in \mathbb{W} approaches the largest invariant set $\mathbb{M} \subset \mathbb{L}$ as $t \rightarrow \infty$ (see [Khalil, 1996]). In our case, by assumption $A^T + A = 0$ and condition (1) of Theorem 4, the role of the set \mathbb{M} can be played either by a limit cycle or by a homoclinic trajectory connected at the saddle point $\mathbf{0}$ (see [Wiggins, 2003]). If both conditions (1) and (3) of Theorem 4 are satisfied, then there exists a sequence of values $r = r_0 > r_{01} > r_{02} > \dots > r_{0k} > \dots$ of parameter r such that $\lim_{k \rightarrow \infty} r_{0k} = r_c \geq 0$ and the set $\mathbb{M}(r_c)$ at the critical value r_c is the homoclinic orbit. (Indeed, let \mathbb{N}_s and \mathbb{N}_u be stable and unstable manifolds of the point $\mathbf{0}$ [Wiggins, 2003]. Let us denote by $\mathbf{x}_0 = (x_{01}, \dots, x_{0n})^T \in \mathbb{N}_u$ the starting point. Since at the point \mathbf{x}_0 we have $\dot{V}_t(x_{01}, \dots, x_{0n}) \geq 0$, then the solution $\mathbf{x}(\mathbf{x}_0, t)$ of system (39) should be attracted to a certain limit cycle in \mathbb{L} . According to condition (3) of Theorem 4, this limit cycle for some value of parameter r will pass arbitrarily close to the origin (to the manifold \mathbb{N}_s). In other words, near the saddle point $\mathbf{0}$ on trajectory $\mathbf{x}(\mathbf{x}_0, t)$ there will be point $\mathbf{x}_1 = (x_{11}, \dots, x_{1n})^T$ such that $\dot{V}_t(x_{11}, \dots, x_{1n}) \leq 0$. Therefore, there must be the value r_c of parameter r for which $\mathbb{N}_s \cap \mathbb{N}_u \neq \emptyset$. This means the existence of homoclinic orbit.)

The last statement enables us to construct a discrete mapping for system (39). To do this, it is necessary to approximate the odd functions (26) by odd power functions (25): $\phi_i(u) < u^{\alpha_i}$, $\psi_i(-u) < (-u)^{\beta_i}$. (A similar approximation must be performed for even functions (27).)

Let us introduce the function

$$\Phi(x_1, \dots, x_n) = \int (x_1)^{\alpha_1 \vee \beta_1} dx_1 \\ + \dots + \int (x_n)^{\alpha_n \vee \beta_n} dx_n,$$

where

$$\int (x_i)^{\alpha_i \vee \beta_i} dx_i = \frac{x_i^{\alpha_i+1}}{\alpha_i+1} \quad \text{if } x_i \geq 0 \quad \text{or} \\ \frac{(-x_i)^{\beta_i+1}}{\beta_i+1} \quad \text{if } x_i < 0; \\ i = 1, \dots, n.$$

It is obvious that

$$V_1(x_1, \dots, x_n) = \int h_1 dx_1 + \dots + \int h_n dx_n \\ \leq \Phi(x_1, \dots, x_n)$$

[see (34)].

Introduce the vectors

$$\overline{\mathbf{h}(\mathbf{x})} = (x_1^{\alpha_1 \vee \beta_1}, \dots, x_n^{\alpha_n \vee \beta_n})^T \quad \text{and} \\ \overline{\mathbf{g}(\mathbf{x})} = (x_1^{\gamma_1 \vee \delta_1}, \dots, x_n^{\gamma_n \vee \delta_n})^T.$$

Now we estimate the function $\dot{V}_{1t}(x_1, \dots, x_n) = \dot{V}_{1t}(x_1(t), \dots, x_n(t)) = \dot{V}_{1t}(t)$ as follows:

$$\dot{V}_{1t}(x_1, \dots, x_n) \leq \dot{\Phi}_t(x_1, \dots, x_n) \\ \leq (\overline{\mathbf{h}(\mathbf{x})}, \overline{\mathbf{g}(\mathbf{x})}) - r \|\overline{\mathbf{h}(\mathbf{x})}\|^2 \\ \leq p_1 \Phi^{(q+s)/(q+1)} - p_0 \Phi^{2q/(q+1)},$$

where $p_0 = p_0(q, s, r) > 0$, $p_1 = p_1(q, s) \geq 0$, and $1 < (q + s)/(q + 1) < 2q/(q + 1) < 2$; $q = \min(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$, $s = \max(\gamma_1, \dots, \gamma_n, \delta_1, \dots, \delta_n)$, and $q > s$.

Now we take advantage of condition (2) of Theorems 4. After that, it is necessary to use the technique of proving Theorem 4.1 [Belozyorov & Dantsev, 2022] and Theorem 1 [Belozyorov, 2019], and construct an appropriate one-dimensional discrete mapping depending on some parameters:

$$\Phi_{k+1} = \Phi_k \exp(p_1 \Phi_k^{(q+s)/(q+1)} - p_0 \Phi_k^{2q/(q+1)}); \\ k = 0, 1, 2, \dots \quad (42)$$

Note that mapping (42) has two fixed points: $\Phi_1^* = 0$ and $\Phi_2^* > 0$, where point Φ_2^* is the root of equation $p_1 - p_0 \Phi^{(q+s)/(q+1)} = 0$.

It can be shown that for some values of the parameters, $\Phi_1^* = 0$ is a repellent point, and point $\Phi_2^* \neq 0$ is an attractive one (this means the existence of a limit cycle in system $\dot{\mathbf{x}}(t) = \overline{\mathbf{g}(\mathbf{x})} + (A - D_0)\overline{\mathbf{h}(\mathbf{x})}$).

Further, let

$$D_0 = 0, \quad \mathbf{g}(\mathbf{x}) \equiv 0.$$

Then Eq. (39) has only periodic solutions. Therefore, the matrix D_0 and the function $\mathbf{g}(\mathbf{x})$ act as generators of nonperiodic processes that may arise after bifurcations of the solutions of equation

$$\dot{\mathbf{x}}(t) = A\mathbf{h}(\mathbf{x}).$$

In addition, in [Belozyorov & Dantsev, 2022] it is also shown that for some $r = r_c$ mapping (42) under the condition $\liminf_{k \rightarrow \infty} \Phi_k = 0$ (Theorem 1 [Belozyorov, 2019]) generates a chaotic attractor. In turn, the fulfillment of the conditions $V_1(x_1, \dots, x_n) \leq \Phi(x_1, \dots, x_n)$ and (41) guarantees the existence of the mentioned parameter r_c (see [Belozyorov, 2017, 2018, 2019], where the function $\rho(t)$ must be replaced by $V_1(t)$). Therefore, system (39) at $r = r_c$ will also exhibit chaotic behavior. ■

Comment. In Theorem 4, the matrix $A - D_0$ can be replaced by a matrix $A - A_s$, where the matrix A is antisymmetric, the matrix A_s is symmetric positive definite, and $r > 0$ is the minimum eigenvalue of matrix A_s . (For applied problems, $A_s = rI$ is usually assumed.)

5.1. Checking the existence of chaos in system (39)

In system (39) we will make the following substitution:

$$\mathbf{g}(\mathbf{x}) \rightarrow B\mathbf{x} + AK\mathbf{g}(\mathbf{x}).$$

Then system (39) will take the following form

$$\dot{\mathbf{x}}(t) = B\mathbf{x} + (A - D_0)\mathbf{h}(\mathbf{x}) + AK\mathbf{g}(\mathbf{x}), \\ \mathbf{h}(\mathbf{x}) > \mathbf{g}(\mathbf{x}) > \mathbf{x}. \quad (43)$$

(In this case, the assertion of Theorem 4 remains valid.) Note that the system (32), which generates periodic and homoclinic orbits, can be considered as a template [Letellier *et al.*, 2006; Veeman *et al.*, 2022], for system (43) ($D_0 \neq 0$), which allows [unlike system (32)] to generate strange chaotic attractors as well.

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(1) Consider the following version of system (43)

$$\begin{cases} \dot{x}(t) = 1.9x + (-rx^5 + y^5 + 3z^5) + (-y^3 + 3z), \\ \dot{y}(t) = -0.4x - y + 2.6z + (-x^5 - ry^5 + z^5) \\ \quad + (x^3 + z), \\ \dot{z}(t) = x - 3y - z + (-3x^5 - y^5 - rz^5) \\ \quad + (3x^3 + y^3). \end{cases} \quad (44)$$

Here $D_0 = rI$, $r \geq 0$, $K = \text{diag}(-1, -1, 1)$,

$$B = \begin{pmatrix} 1.9 & 0 & 0 \\ -0.4 & -1 & 2.6 \\ 1 & -3 & -1 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -3 & -1 & 0 \end{pmatrix},$$

$$\mathbf{h}(\mathbf{x}) = (x^5, y^5, z^5)^T,$$

$$AK\mathbf{g}(\mathbf{x}) = (-y^3 + 3z, x^3 + z, 3x^3 + y^3)^T.$$

Figure 6 shows the attractors of system (44) for various values of r .

Figure 7 shows the results of the verification of Theorem 4 for system (44). (Here the equilibrium 0 is a saddle point. The Jacobi matrix at this point has the following eigenvalues: 2.3188, $-0.2094 \pm 2.7798i$.)

(Note that system (44) is topologically close to the system presented in paper [Wu *et al.*, 2021].)

(2) Now consider the system

$$\begin{cases} \dot{x}(t) = -2x + 10yz - ax^3, \\ \dot{y}(t) = 3xz - by^3, \\ \dot{z}(t) = 3z - xy - cz^3, \end{cases} \quad (45)$$

where $a \geq 0$, $b \geq 0$, and $c \geq 0$. (A detailed analysis of the dynamic behavior of system (45) at $a = 0$, $b = 6$, and $c = 0$ was investigated by Zhang (see reference in article [Benkouider *et al.*, 2021]).)

Let us set

$$V_1(t) = 0.5(x^2(t) + y^2(t) + 13z^2(t)).$$

Then for system (45) we have

$$\begin{aligned} \dot{V}_1(t) = & -2x^2(t) + 39z^2(t) - ax^4(t) \\ & - by^4(t) - 13cz^4(t). \end{aligned}$$

(We could choose function $(x^6(t) + y^6(t) + z^6(t))/6$ as the function $V_1(t)$. However, in this case, the study of system (45) would be more complicated.)

5.2. Constructive way of designing nonstrange chaotic attractors based on the concept of antisymmetry

Let us go back to system (32). We also assume that the conditions of Theorem 3 are satisfied for this system.

In order to consider a specific example of designing a chaotic system, we assume that $n = 4$. In addition, we introduce the following odd functions: $\mathbf{g}(\mathbf{x}) = (x, y, z, u)^T$ and $\mathbf{h}(\mathbf{x}) = (x \cdot |x|, y \cdot |y|, z \cdot |z|, u \cdot |u|)^T$. The invertible antisymmetric matrix A

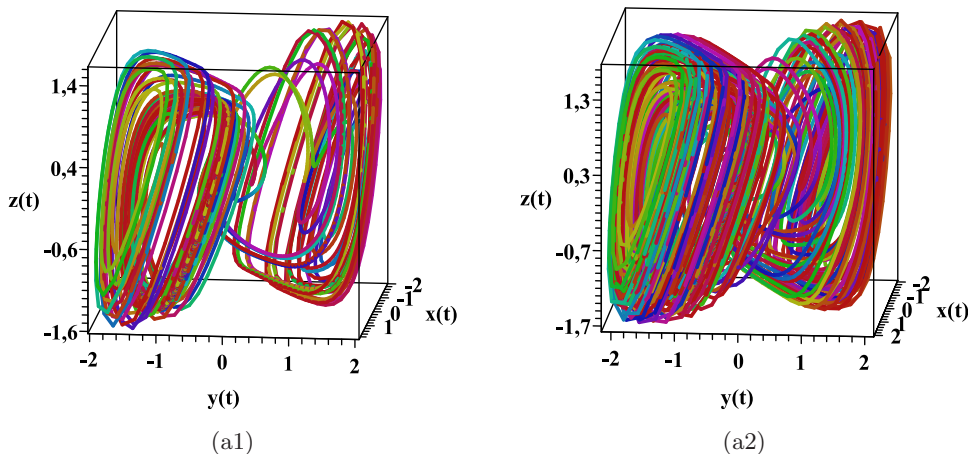


Fig. 6. Attractors generated by system (44) for various values of r : (a1) $r = 0.01$ and (a2) $r = 0$.

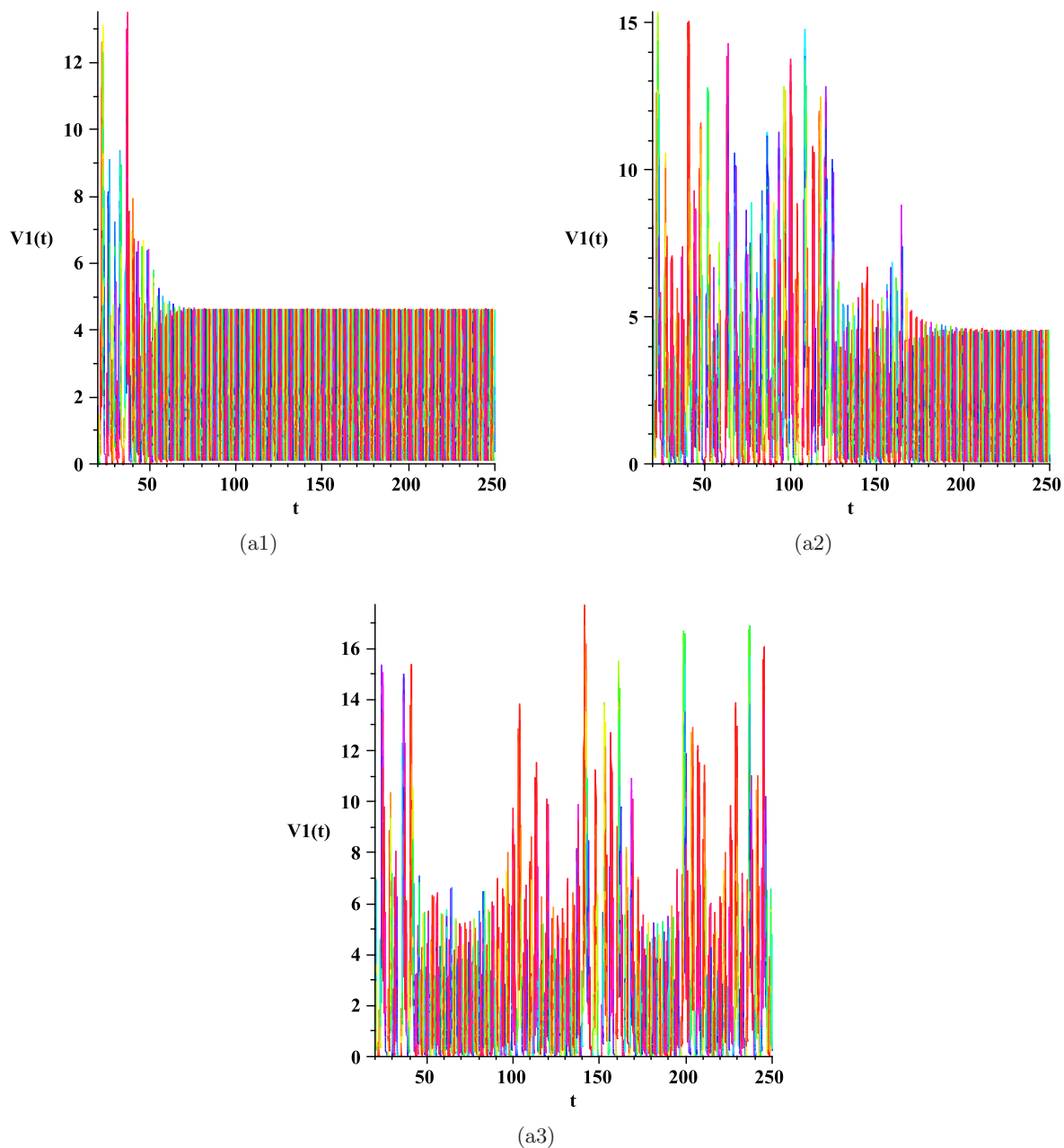


Fig. 7. The behavior of the function $V_1(t) = (x^6(t) + y^6(t) + z^6(t))/6$ for system (44) at $r = 0.01$ (a1) (the limit cycle; $\liminf_{t \rightarrow \infty} V_1(t) \neq 0$), $r = 0.001$ (a2) (the limit cycle; $\liminf_{t \rightarrow \infty} V_1(t) \neq 0$) and $r = 0$ (a3) (the chaotic attractor; $\liminf_{t \rightarrow \infty} V_1(t) = 0$).

can be chosen in an arbitrary form, for example:

$$A = \begin{pmatrix} 0 & -3 & 1 & 1 \\ 3 & 0 & 2 & 0 \\ -1 & -2 & 0 & 4 \\ -1 & 0 & -4 & 0 \end{pmatrix}.$$

It can be shown that if $\mathbf{g}(\mathbf{x}) = (x, y, z, u)^T$, then the Jacobian matrix of system (32) at the point $\mathbf{0}$ has the form AK . Therefore, in case $n = 4$, the

characteristic polynomial of such matrix is

$$q(\lambda) = \lambda^4 + a_2\lambda^2 + a_4,$$

where $a_4 \neq 0$. Depending on the coefficients a_2 and a_4 there are four different sets of roots for this polynomial: (1) $q(\lambda) = \lambda^4 + a_2\lambda^2 + a_4 = (\lambda^2 + p_1)(\lambda^2 + p_2)$, $p_1 > 0$, $p_2 > 0$ and all roots are purely imaginary; (2) $q(\lambda) = \lambda^4 + a_2\lambda^2 + a_4 = (\lambda^2 + p_1)(\lambda^2 + p_2)$, $p_1 > 0$, $p_2 < 0$ and two real conjugate roots and two purely imaginary roots;

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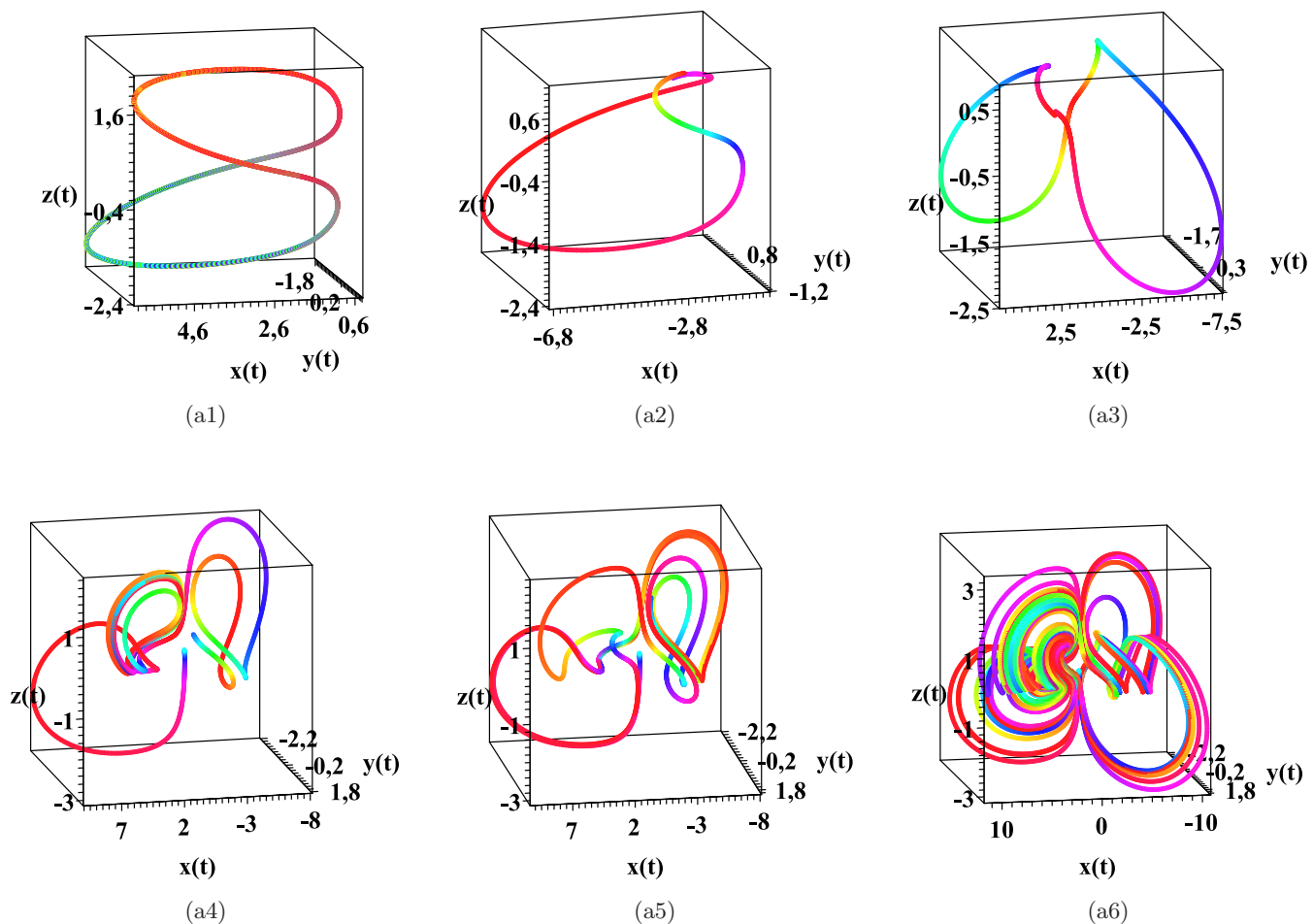


Fig. 8. The dynamics of system (45) for different values of parameters a and c ($b = 6$). Limit cycle: (a1) $a = 0.1$, $c = 0.02$; (a2) $a = 0.05$, $c = 0.02$; (a3) $a = 0.03$, $c = 0.02$. Homoclinic orbit: (a4) $a = 0.01$, $c = 0.01$; (a5) $a = 0.005$, $c = 0.01$. Chaotic behavior: (a6) $a = 0.00$, $c = 0.00$.

(3) $q(\lambda) = \lambda^4 + a_2\lambda^2 + a_4 = (\lambda^2 + p_1)(\lambda^2 + p_2)$, $p_1 < 0$, $p_2 < 0$ and four real pairwise conjugate roots (all these cases are shown respectively in Figs. 10–12); (4) $q(\lambda) = \lambda^4 + a_2\lambda^2 + a_4 = (\lambda^2 + p + is)(\lambda^2 + p - is)$, $s \neq 0$ and all roots are complex conjugate.

We introduce a positive definite function [see (34)]

$$\begin{aligned}
 V_1(x, y, z, u) &= \int_0^x s_1 |s_1| ds_1 + \int_0^y s_2 |s_2| ds_2 \\
 &\quad + \int_0^z s_3 |s_3| ds_3 + \int_0^u s_4 |s_4| ds_4 \\
 &= \frac{1}{3}(x^2|x| + y^2|y| + z^2|z| + u^2|u|), \\
 V_{10} &= \frac{1}{3}(x_0^2|x_0| + y_0^2|y_0| + z_0^2|z_0| + u_0^2|u_0|).
 \end{aligned}$$

From here it follows that

$$\begin{aligned}
 \dot{V}_{1t} &= \frac{\partial V_1}{\partial x} \dot{x}_t + \frac{\partial V_1}{\partial y} \dot{y}_t + \frac{\partial V_1}{\partial z} \dot{z}_t + \frac{\partial V_1}{\partial u} \dot{u}_t \\
 &= x|x|\dot{x}_t + y|y|\dot{y}_t + z|z|\dot{z}_t + u|u|\dot{u}_t,
 \end{aligned}$$

where the derivatives are determined by virtue of the equations of system (32). (It should be noted that the derivative \dot{V}_{1t} of the function $V_1(x, y, z, u)$ was calculated separately for intervals $t \geq 0$ and $t < 0$.)

Let us now consider the behavior of the solutions of system (32) with the given matrix A and different matrices K (see Figs. 10–12).

We put in Theorem 4 $\mathbf{g}(\mathbf{x}) \rightarrow A\mathbf{g}(\mathbf{x})$, $B = 0$, and $D_0 = 0$. Now it can be applied to system (32) (see Fig. 13).

Let $\Lambda_1 \geq \dots \geq \Lambda_n$ be the Lyapunov exponents for a dynamical system in \mathbb{R}^n . Assume that j is the

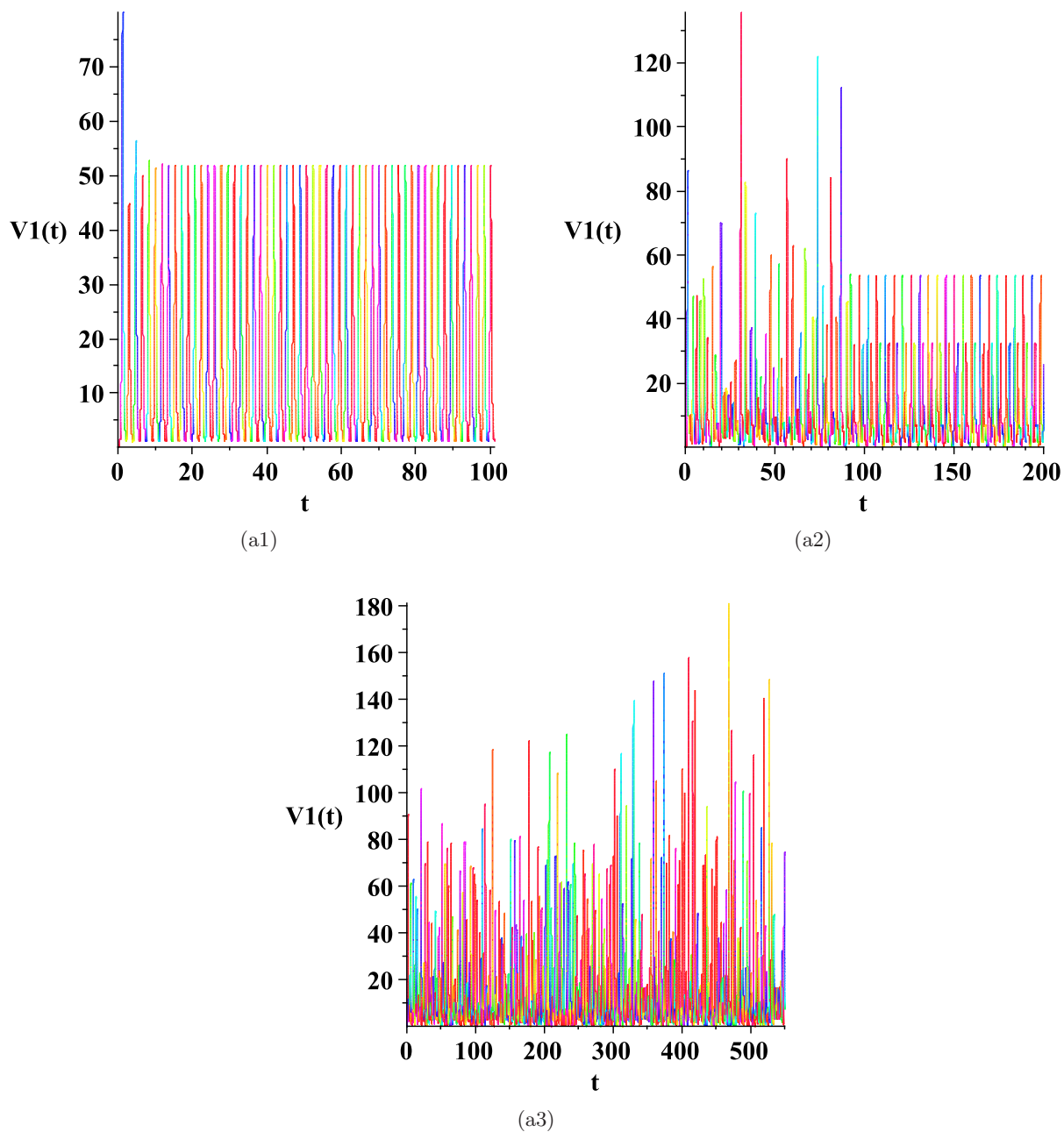


Fig. 9. The behavior of the function $V_1(t) = (x^2(t) + y^2(t) + z^2(t))/2$ for system (45) at (a1) $a = 0.1$, $b = 6$, $c = 0.02$; (a2) $a = 0.01$, $b = 6$, $c = 0.01$ and (a3) $a = 0.0$, $b = 6$, $c = 0.0$.

largest integer for which $\Lambda_1 + \dots + \Lambda_j \geq 0$. In this case, the Kaplan–Yorke dimension is given by the formula (see [Banks, 1999]):

$$d_{KL} = j + \frac{\Lambda_1 + \dots + \Lambda_j}{|\Lambda_{j+1}|}.$$

Let $n = 4$. Then we can show (see [Belozyrov *et al.*, 2021]) that the Lyapunov exponents for system (32) can be ordered as follows: $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3 = -\Lambda_2 \geq \Lambda_4 = -\Lambda_1$. Thus, the Kaplan–Yorke dimension is $d_{KL} = 4$ and coincides with

the usual dimension of the Euclidean space for system (32). This means that the considered system has a nonstrange attractor, which is chaotic only in case $\Lambda_1 > 0$. (The last condition is achieved only in the case $K \neq \text{diag}(1, 1, 1, 1)$. If $K = \text{diag}(1, 1, 1, 1)$, then $\Lambda_1 = \dots = \Lambda_4 = 0$.)

Comment. It is easy to verify that all the results obtained in Sec. 5 remain valid if the matrix $K = \text{diag}[(-1)^{k_1}, \dots, (-1)^{k_4}]$ is replaced by the matrix $K_0 = \text{diag}[k_{11}, \dots, k_{44}]$, where $\det K_0 \neq 0$. Indeed,

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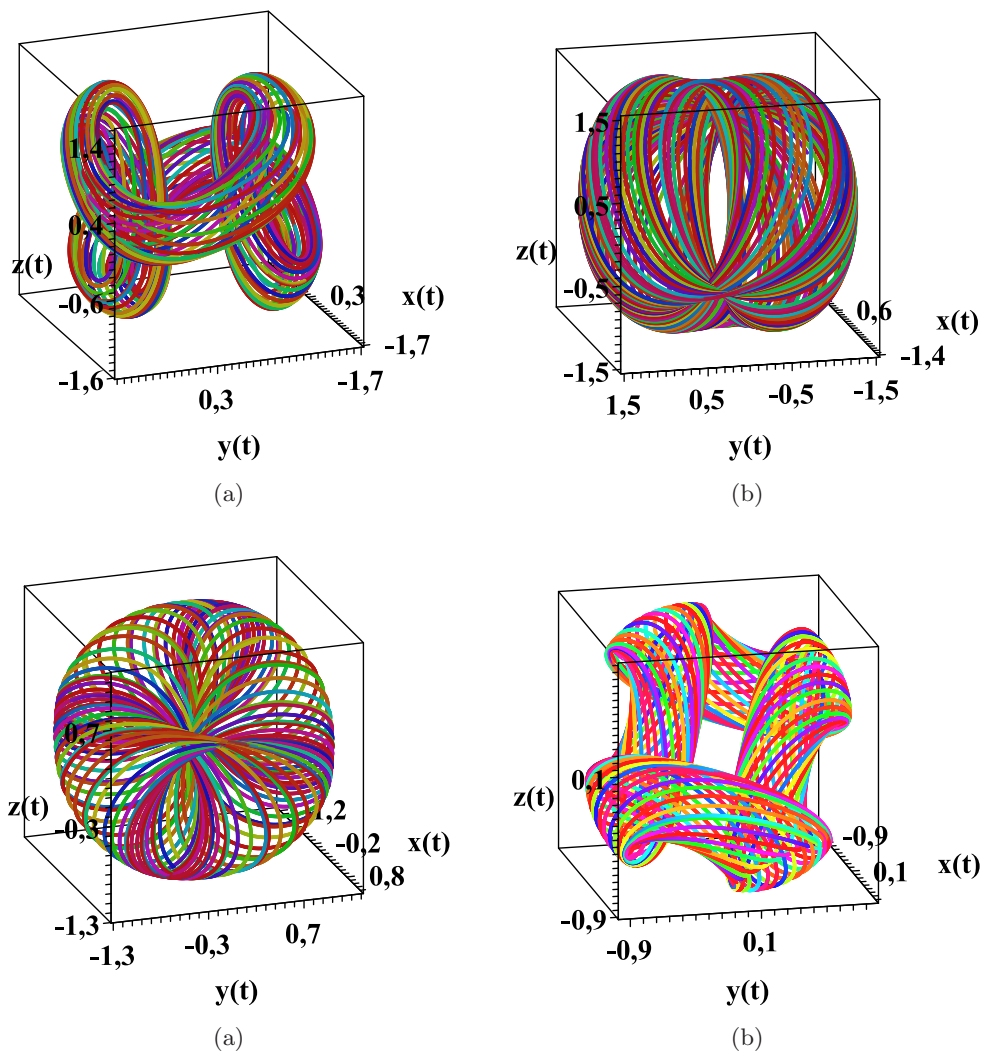


Fig. 10. The behavior of the trajectories of system (32) (3D projection) for $K = \text{diag}(1, 1, 1, 1)$ and various initial conditions. The Jacobian matrix at point $\mathbf{0}$ has a characteristic polynomial $q(\lambda) = \lambda^4 + 31\lambda^2 + 100$, whose roots $\pm 5.22903i, \mp 1.91240i$ are purely imaginary. All solutions of this system are periodic; there are no homoclinic orbits.

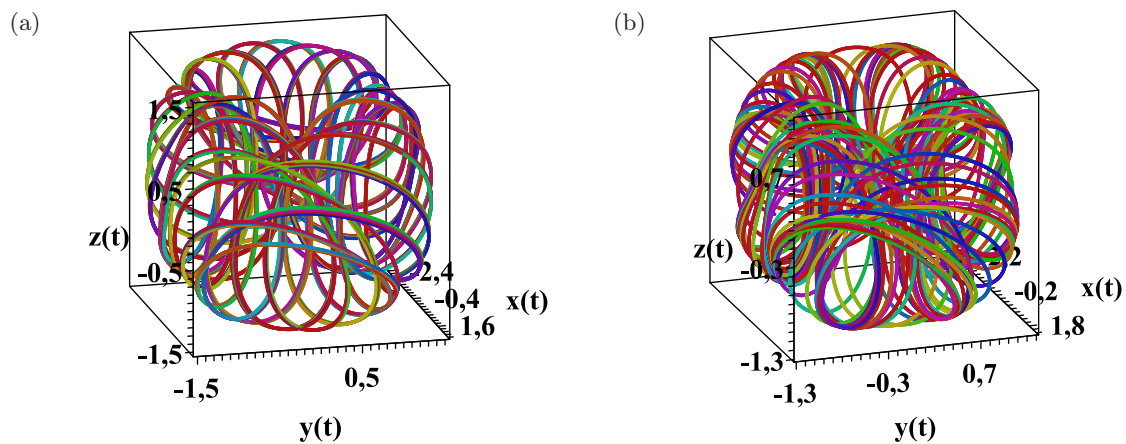


Fig. 11. The behavior of the trajectories of system (32) (3D projection) for $K = \text{diag}(-1, 1, 1, 1)$ and various initial conditions. The Jacobian matrix at point $\mathbf{0}$ has a characteristic polynomial $q(\lambda) = \lambda^4 + 9\lambda^2 - 100$. It has two real and two purely imaginary roots: $\pm 2.54280, \mp 3.93267i$. There are homoclinic orbits. However, for given parameters, there is no chaos in system (32).

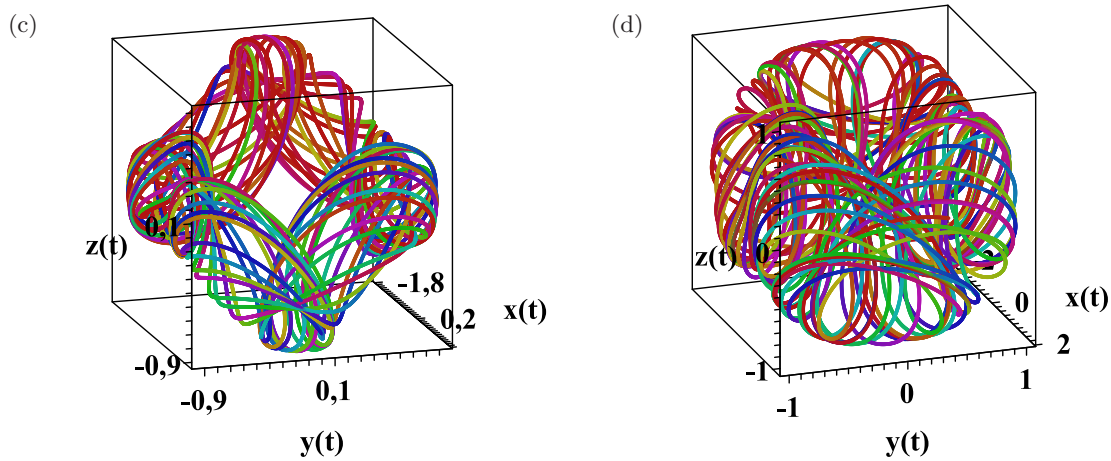


Fig. 11. (Continued)

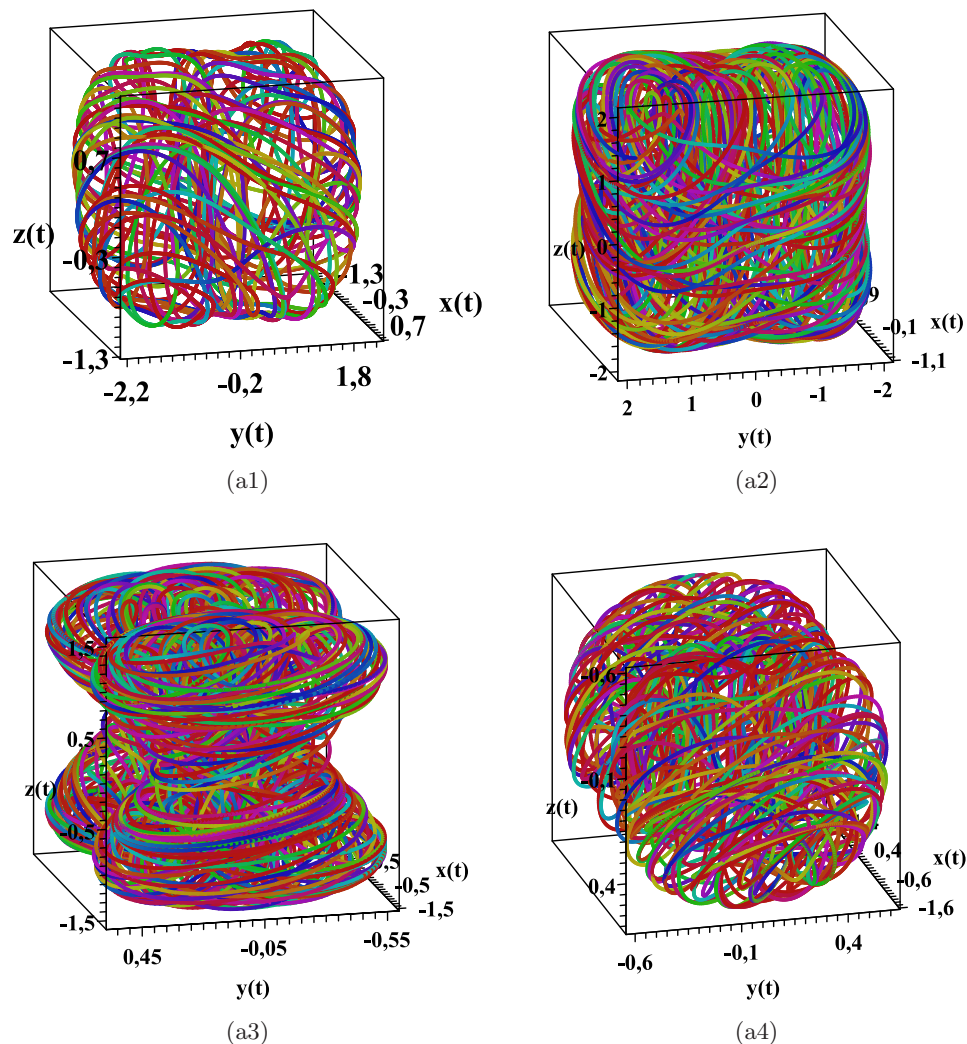


Fig. 12. The behavior of the trajectories of system (32) (3D projection): (a1) $K = \text{diag}(1, -1, 1, -1)$, characteristic polynomial $q(\lambda) = \lambda^4 - 29\lambda^2 + 100$ of the Jacobian matrix in point $\mathbf{0}$ and its roots $\pm 2, \mp 5$; (a2) $K = \text{diag}(1, -1, -1, 1)$, characteristic polynomial $q(\lambda) = \lambda^4 - 21\lambda^2 + 100$ of the Jacobian matrix in point $\mathbf{0}$ and its roots $\pm 2.70156, \mp 3.70156$; (a3) $K = \text{diag}(-1, 1, -1, 1)$, characteristic polynomial and its roots are the same as in case (a1); (a4) $K = \text{diag}(-1, 1, 1, -1)$, characteristic polynomial and its roots are the same as in case (a2). There are homoclinic orbits and chaos (see Fig. 13).

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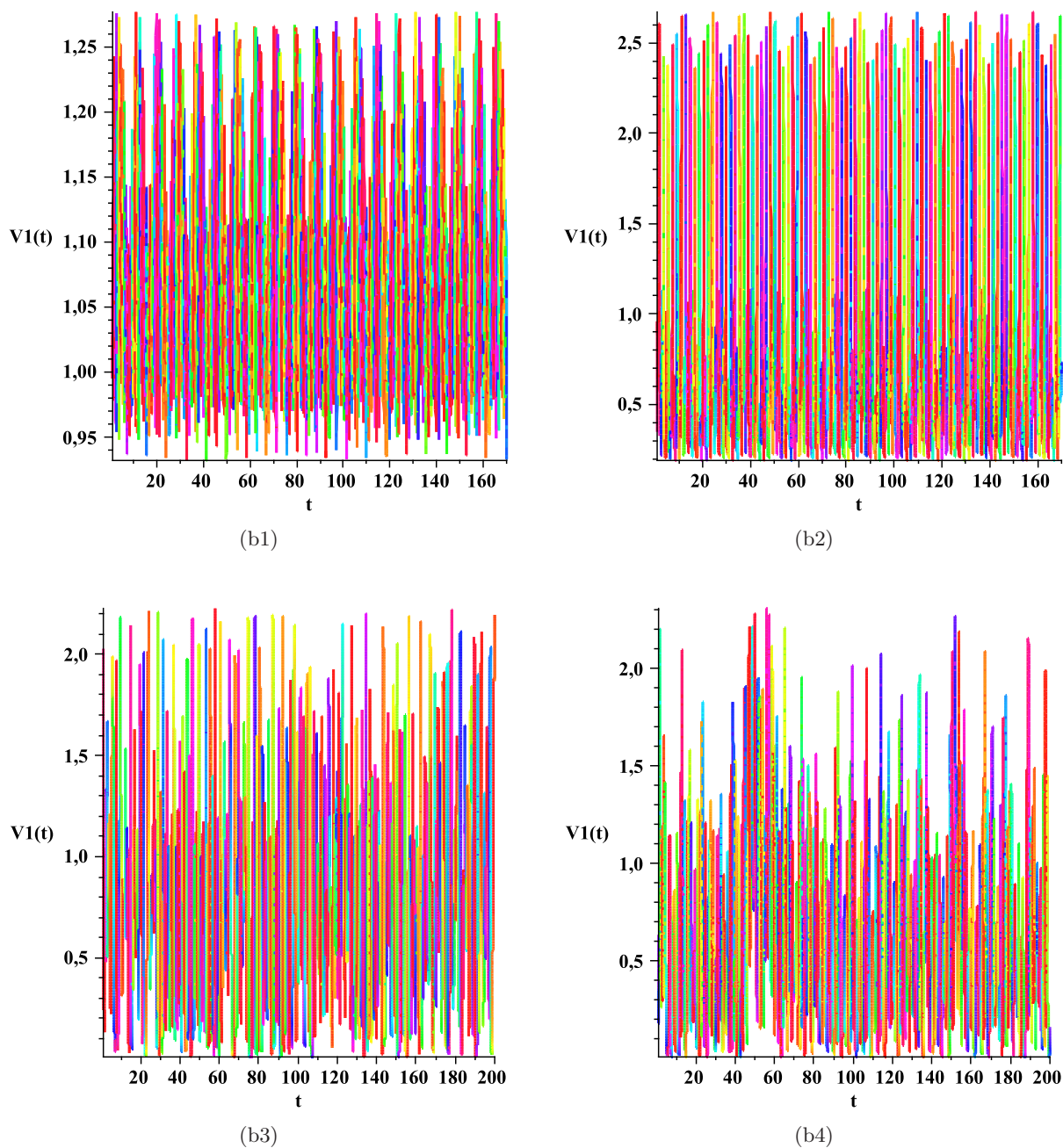


Fig. 13. The function $V_1(t)$ from Theorem 4: (b1) $K = \text{diag}(1, 1, 1, 1)$ (see Fig. 10) (there is no chaos) and (b2) $K = \text{diag}(-1, 1, 1, 1)$ (see Fig. 11) (there is no chaos). The same function $V_1(t)$: (b3) $K = \text{diag}(-1, 1, -1, 1)$ and (b4) $K = \text{diag}(-1, 1, 1, -1)$ [see Figs. 12(a3) and 12(a4)]: there is chaos.

in this case, the characteristic polynomial $q(\lambda) = \lambda^4 + r_2\lambda^2 + r_4$ of matrix AK_0 retains the form of characteristic polynomial $q(\lambda) = \lambda^4 + a_2\lambda^2 + a_4$ of matrix AK .

6. Conclusion

New concepts of even and odd functions, generalizing the well-known classical definitions of similar

functions, are introduced (see [Belozyorov & Dantsev, 2020, 2022; Belozyorov *et al.*, 2021]). Based on these concepts, a new method for constructing chaotic attractors of antisymmetric neural ODEs is proposed. The essence of the method is the introduction of system (32), which is guaranteed to contain a homoclinic orbit. Further, system (32) is used as the template for system (43), which can generate an arbitrary number of chaotic attractors.

(However, as shown in Fig. 13(b2), the presence of the homoclinic orbit does not always lead to chaos in system (32).)

References

- Banks, S. P. [1999] “Analysis and structure of a new chaotic system,” *Int. J. Bifurcation and Chaos* **9**, 1571–1583.
- Belozyorov, V. Ye. [2011a] “On existence of homoclinic orbits for some types of autonomous quadratic systems of differential equations,” *Appl. Math. Comp.* **217**, 4582–4595.
- Belozyorov, V. Ye. [2011b] “New types of 3D systems of quadratic differential equations with chaotic dynamics based on Ricker discrete population mode,” *Appl. Math. Comp.* **218**, 4546–4566.
- Belozyorov, V. Ye. [2014] “General method of construction of implicit discrete maps generating chaos in 3D quadratic systems of differential equations,” *Int. J. Bifurcation and Chaos* **24**, 1450025.
- Belozyorov, V. Ye. [2015a] “Exponential-algebraic maps and chaos in 3D autonomous quadratic systems,” *Int. J. Bifurcation and Chaos* **25**, 1550048.
- Belozyorov, V. Ye. [2015b] “Research of chaotic dynamics of 3D autonomous quadratic systems by their reduction to special 2D quadratic systems,” *Math. Probl. Engin.* **2015**, 271637.
- Belozyorov, V. Ye. [2016] “A novel search method of chaotic autonomous quadratic dynamical systems without equilibrium points,” *Nonlin. Dyn.* **86**, 835–860.
- Belozyorov, V. Ye. [2017] “Reduction method for search of chaotic attractors in generic autonomous quadratic dynamical systems,” *Int. J. Bifurcation and Chaos* **27**, 1750036.
- Belozyorov, V. Ye. [2018] “On novel conditions of chaotic attractors existence in autonomous polynomial dynamical systems,” *Nonlin. Dyn.* **91**, 2435–2452.
- Belozyorov, V. Ye. [2019] “Universal approach to the problem of emergence of chaos in autonomous dynamical systems,” *Nonlin. Dyn.* **95**, 579–595.
- Belozyorov, V. Ye. & Dantsev, D. V. [2020] “Stability of neural ordinary differential equations with power nonlinearities,” *J. Optimiz. Diff. Eqs. Their Appl. (JODEA)* **28**, 21–46.
- Belozyorov, V. Ye., Dantsev, D. V. & Koshel, Y. V. [2021] “On the equivalence of real dynamic process and its models,” *J. Optimiz. Diff. Eqs. Their Appl. (JODEA)* **29**, 76–91.
- Belozyorov, V. Ye. & Dantsev, D. V. [2022] “Modeling of chaotic processes by means of antisymmetric neural ODEs,” *J. Optimiz. Diff. Eqs. Their Appl. (JODEA)* **30**, 1–41.
- Benkouider, K., Bouden, T., Sambas, A., Mohamed, M. A., Sulaiman, I. M., Mamat, M. & Ibrahim, M. A. H. [2021] “Dynamics, control and secure transmission electronic circuit implementation of a new 3D chaotic system in comparison with 50 reported systems,” *IEEE Access* **9**, 152150–152168.
- Chang, B., Chen, M., Haber, E. & Chi, E. D. [2019] “Antisymmetric RNN: A dynamical system view on recurrent neural networks,” In *Conference ICLR*, May 6–9, New Orleans, Louisiana, USA, pp. 1–15.
- Chen, R. T. Q., Rubanova, Y., Bettencourt, J. & Duvenaud, D. [2019] “Neural ordinary differential equations,” 1–18, arXiv:1806.07366v5[cs.LG].
- Goodfellow, I., Bengio, Y. & Courville, A. [2017] *Deep Learning* (The MIT Press Cambridge, Massachusetts, London, England).
- Gu, S., Peng, Q., Leng, X. & Du, B. [2021] “A novel non-equilibrium memristor-based system with multi-wing attractors and multiple transient transitions,” *Chaos* **31**, 033105.
- Haber, E. & Ruthotto, L. [2019] “Stable architectures for deep neural networks,” 1–23, arXiv:1705.03341v3[cs.LG].
- Hu, G. [2009] “Generating hyperchaotic attractors with three positive Lyapunov exponents via state feedback control,” *Int. J. Bifurcation and Chaos* **19**, 651–660.
- Kai, G., Zhang, W., Wei, Z. C., Wang, J. F. & Akgul, A. [2017] “Hopf bifurcation, positively invariant set, and physical realization of a new four-dimensional hyperchaotic financial system,” *Math. Probl. Engin.* **2017**, 2490580.
- Khalil, H. K. [1996] *Nonlinear Systems*, 2nd edition (Prentice Hall, Upper Saddle River).
- Lai, Q., Akgul, A., Li, C., Xu, G. & Çavuşoğlu, U. [2018] “A new chaotic system with multiple attractors: Dynamic analysis, circuit realization and S-box design,” *Entropy* **20**, 1–15.
- Lai, Q. [2020] “A unified chaotic system with various coexisting attractors,” *Int. J. Bifurcation and Chaos* **31**, 2150013.
- Lai, Q., Wan, Z., Akgul, A., Boyraz, O. F. & Yildiz, M. Z. [2020a] “Design and implementation of a new memristive chaotic system with application in touchless fingerprint encryption,” *Chinese J. Phys.* **67**, 615–630.
- Lai, Q., Kuate, P. D. K., Pei, H. & Fotsin, H. [2020b] “Infinitely many coexisting attractors in no-equilibrium chaotic system,” *Complexity* **2020**, 8175639-1–17.
- Letellier, C., Roulin, E. & Rossler, O. E. [2006] “Inequivalent topologies of chaos in simple equations,” *Chaos Solit. Fract.* **28**, 337–360.
- Li, Q. [2020] *Dynamical Systems and Machine Learning* (Summer School, Peking University).
- Liu, X., Wang, J. & Huang, L. [2007] “Attractors of fourth-order Chua’s circuit and chaos control,” *Int. J. Bifurcation and Chaos* **17**, 2705–2722.

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- Liu, M., Feng, J. & Tse, C. K. [2010] “A new hyperchaotic system and its circuit implementation,” *Int. J. Bifurcation and Chaos* **20**, 1201–1208.
- Liu, L., Du, C., Liang, L. & Zhang, X. [2019] “A high spectral entropy (SE) memristive hidden chaotic system with multi-type quasi-periodic and its circuit,” *Entropy* **21**, 1–24.
- Lu, J. & Chen, G. [2006] “Generating multiscroll chaotic attractors: Theories, methods and applications,” *Int. J. Bifurcation and Chaos* **16**, 775–858.
- Pham, P.-T. & Hong, K.-S. [2020] “Dynamic models of axially moving systems: A review,” *Nonlin. Dyn.* **100**, 315–349.
- Sooraksa, P. & Chen, G. [2018] “Chen system as a controlled weather model — Physical principle, engineering design and real applications,” *Int. J. Bifurcation and Chaos* **28**, 1830009.
- Veeman, D., Natiq, H., Ali Ali, A., Rajagopal, K. & Hussain, I. [2022] “A simple conservative chaotic oscillator with line of equilibria: Bifurcation plot, basin analysis, and multistability,” *Complexity* **2022**, 9345036.
- Wang, J., Chen, Z., Chen, G. & Yuan, Z. [2008] “A novel hyperchaotic system and its complex dynamics,” *Int. J. Bifurcation and Chaos* **18**, 3309–3324.
- Wang, J., Chen, Z. & Yuan, Z. [2009] “Existence of a new three-dimensional chaotic attractor,” *Chaos Solit. Fract.* **42**, 3053–3057.
- Wang, X. & Chen, G. [2013] “A gallery Lorenz-like and Chen-like attractors,” *Int. J. Bifurcation and Chaos* **23**, 1330011.
- Wang, Z., Moroz, I., Wei, Z. & Ren, H. [2018] “Dynamics at infinity and a Hopf bifurcation arising in a quadratic system with coexisting attractors,” *Pramana — J. Phys.* **90**, 1–10.
- Wang, X., Kuznetsov, N. V. & Chen, G. [2021] *Chaotic Systems with Multistability and Hidden Attractors* (Springer, Cham).
- Wei, Z., Moroz, I., Sprott, J. C., Akgul, A. & Zhang, W. [2017] “Hidden hyperchaos and electronic circuit application in a 5D self-exciting homopolar disc dynamo,” *Chaos* **27**, 033101.
- Wiggins, S. [2003] *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, 2nd edition (Springer-Verlag, NY, Berlin, Heidelberg).
- Wu, X., Wang, H. & He, S. [2021] “Localization of hidden attractors in Chua’s system with absolute nonlinearity and its FPGA implementation,” *Front. Phys.* **9**, 788329.
- Yang, Q. & Chen, G. [2008] “A chaotic system with one saddle and two stable node-foci,” *Int. J. Bifurcation and Chaos* **18**, 1393–1414.
- Yang, Q., Wei, Z. & Chen, G. [2010] “An unusual 3D autonomous quadratic chaotic system with two stable node-foci,” *Int. J. Bifurcation and Chaos* **20**, 1061–1083.
- Yu, S., Tang, W. K. S., Lu, J. & Chen, G. [2010] “Design and implementation of multi-wing butterfly chaotic attractors via Lorenz-type systems,” *Int. J. Bifurcation and Chaos* **20**, 29–41.
- Yu, B. & Hu, G. [2010] “Constructing multiwing hyperchaotic attractors,” *Int. J. Bifurcation and Chaos* **20**, 727–734.
- Zhou, L., Chen, Z., Wang, J. & Zhang, Q. [2017] “Local bifurcation analysis and global dynamics estimation of a novel 4-dimensional hyperchaotic system,” *Int. J. Bifurcation and Chaos* **27**, 1750021.
- Zhu, X. & Du, W.-S. [2019] “A new family of chaotic systems with different closed curve equilibrium,” *Mathematics* **7**, 1–8.