Roman Voliansky, Vitaliy Kuznetsov, Volodymyr Doskoch, Oleksandr Tkalenko

SCIENTIFIC BASIS OF ANALYSIS AND SYNTHESIS OF OPTIMAL ENERGY-EFFICIENT CONTROL OF ELECTROMECHANICAL SYSTEMS

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The authors' achievements in the field of development of control systems theory, where high-order sliding modes are implemented, was presented. Original methods for optimizing the accuracy of automatic ¬control systems for electric drives with nonlinear activation functions were outlined. The dynamic and static characteristics of control systems with nonlinear control algorithms were considered, and their stability was analyzed.
For scientific and technical workers engaged in the research, design and operation of modern automatic control systems, as well as for students and post-graduate students of relevant specialties.

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INTRODUCTION

Currently, the problem of energy conservation is quite acute for industry. This problem is being addressed at different levels through the use of a series of measures aimed at both creating fundamentally new approaches to the technological processes management and modernizing existing ones.

One of these measures is energy conservation through the means of electric drives. Electrical energy savings can be guaranteed by using energy-efficient laws of optimal control for electromechanical systems.

The synthesis of optimal control systems, based on solving the problem of analytical design of controllers using known methods, is carried out using linear or linearized differential equations of perturbed motion. The advantages of this approach include the transparency of the mathematical apparatus used at the stage of creating a control system, and the simplicity of the technical implementation of the synthesized system. However, along with the advantages, this approach has a number of disadvantages. The main one is that synthesized discontinuous or linear controls leading to a change in the coordinates of the perturbed motion according to an exponential law, are not optimal from an energy point of view. Furthermore, the use of discontinuous controls leads to significant overvoltages on the power switches of the converter during switching and can lead to premature aging or destruction of the power circuit. Known measures to eliminate these drawbacks and improve the dynamic performance of an electric drive in real-time are subjective and do not have a strict mathematical justification.

Therefore, recent scientific literature has paid considerable attention to high-order sliding modes. The control actions that implement them can ensure high performance and accuracy of classical systems, but without the switching of the control element. By eliminating overvoltages in the power circuit, these electromechanical systems have increased reliability compared to known relay systems, and by limiting the supply voltage at any intermediate level, improved energy performance can be achieved.

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Based on the above, an urgent problem arises in the development and research of electromechanical systems with high-order sliding modes.

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CHAPTER 1
METHODS OF ANALYSIS OF ELECTRO-MECHANICAL SYSTEMS

1.1. Mathematical description of dynamic systems

When analyzing and synthesizing control systems for technical objects, one usually deals with their mathematical models. A mathematical model is an equation or system of equations that describes the processes occurring in the object under study. This model can be obtained either experimentally or analytically, based on physical, chemical and other laws governing the processes in the considered object.

Mathematical descriptions are based on contradictory requirements. On the one hand, the mathematical model should reflect the properties of the original system as accurately as possible, and on the other hand, it should be as simple as possible to avoid complicating the research. Therefore, at the initial stage of the research, a simpler model should be used, and then, if necessary, the model can be made more complex, taking into account additional factors that were not considered at the initial stage.

1.1.1. Equation of dynamics and statics

Any control system as a whole and its individual elements convert the input signal $x(t)$ into an output signal $y(t)$. Mathematically, they carry out a transformation of the following form

$$y(t) = Ax(t), \tag{1.1}$$

in which each element $x(t)$ from the set of input signals corresponds to a certain well-defined element $y(t)$ from the set of output signals. In the above relationship A is called an operator. The operator that defines the mapping between the input and output signals of a control system or its element is called the operator of this system (or element). To define a system operator means to specify the rule by which the system’s output signal is determined by its input signal.

In the following discussion, we will consider systems, whose operators can be specified using ordinary differential equations.

A mathematical model can be presented as a structure consisting of dynamic links. A link is a mathematical model of a system or part of it, which is defined by certain operator. In a particular case, a link can represent a mathematical model of an element.

The movement of a generalized link can be defined as follows:

$$F(y, py, p^2y\ldots, p^n y, u, pu, p^2u\ldots, p^m u) = 0, \qquad (1.2)$$

where y is an output variable; u is an input variable; $p = \frac{d}{dt}$ is a differentiation operator.

The equation (1.2), which describes the processes in a link under arbitrary input influences, is called the dynamics equation.

From the equation (1.2), a statics equation can be obtained that describes the steady-state mode with constant values of the output and input variables:

$$py = p^2y = p^n y = pu = p^2u = p^m u = 0; \qquad (1.3)$$

$$F_0(y_0, 0, 0\ldots 0, u_0, 0, 0\ldots, 0) = 0. \qquad (1.4)$$

In the general case, when a link is described by a differential equation, the value of the output variable at a moment in time t depends on the previous history, i.e. it depends on the values of the output variable y till the moment t. In this case we talk about a dynamic link.

If a link is described by a function, that is, its value in the output variable at any time depends only on the values of the input variable at the same time, then the static equation coincides with the dynamics equation, and the link is static.

The static mode can be graphically described using a static characteristic. The static characteristic of a link (or element) is the dependence of the output variable on the input variable in a steady-state mode. The static characteristic of an element can be determined experimentally by applying constant influences to its input and measuring the values of the output variable after the end of the transient process, or calculated using the statics equation.

1.1.2. Equations of dynamics of control systems in normal form

If the equation of the system is represented by the highest derivative of the output variable of n-th order, then it can always be turned into a system of n equations of the first order.

The system dynamics equation

$$py^n = F(y, py, \ldots, p^{n-1}y, u), \tag{1.5}$$

by introduction of $y_1 = y;\, y_2 = py, \ldots y_n = py^{n-1}$, is transformed into the form

$$py_1 = y_2; \quad py_2 = y_3; \quad \cdots \quad py_n = F(y_1, y_2 \ldots, y_n, u). \tag{1.6}$$

A similar transformation can be performed when the system is described by several equations of the form

$$\begin{aligned} &p^3 y_1 = F_1(y_1, py_1, p^2 y_1, y_2, py_2, u); \\ &p^2 y_2 = F_2(y_1, py_1, p^2 y_1, y_2, py_2, u). \end{aligned} \tag{1.7}$$

Equations (1.7) can be transformed into a system of first-order equations:

$$\begin{aligned} &px_1 = x_2;\; px_2 = x_3;\; px_3 = F_1(x_1, x_2, x_3, x_4, x_5, u); \\ &px_4 = x_5;\; px_5 = F_2(x_1, x_2, x_3, x_4, x_5, u), \end{aligned} \tag{1.8}$$

where $x_1 = y_1;\, x_2 = py_1;\, x_3 = p^2 y_1;\, x_4 = y_2;\, x_5 = py_2$.

In general, the dynamics equation of the controllable system can be represented in the following form

$$\begin{aligned} &px_1 = f_1(x_1, x_2, \ldots, x_n, u_1, u_2 \ldots, u_r, t); \\ &px_2 = f_2(x_1, x_2, \ldots, x_n, u_1, u_2 \ldots, u_r, t); \\ &\vdots \\ &px_n = f_n(x_1, x_2, \ldots, x_n, u_1, u_2 \ldots, u_r, t); \\ &y_1 = h_1(x_1, x_2, \ldots, x_n, u_1, u_2 \ldots, u_r, t); \\ &y_2 = h_2(x_1, x_2, \ldots, x_n, u_1, u_2 \ldots, u_r, t); \\ &\vdots \\ &y_m = h_n(x_1, x_2, \ldots, x_n, u_1, u_2 \ldots, u_r, t), \end{aligned} \tag{1.9}$$

where n is the number of state variables; m – the number of observable variables; $x_1, x_2, \ldots, x_n$ – phase coordinates; $u_1, u_2 \ldots, u_r$ – control inputs; $y_1, y_2 \ldots, y_m$ – output variables; t – time.

First-order differential equations, written with respect to the first derivative of i -th state variable, are called *a system of differential equations in the normal Cauchy form* or simply a system in the normal form.

In the vector form, the above equations take the form

$$p\mathbf{x} = \mathbf{f}(\mathbf{x},\mathbf{u},t); \quad (1.10)$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x},\mathbf{u},t). \quad (1.11)$$

Here $\mathbf{x}$ it is called *a phase vector* or *a state vector*, $\mathbf{u}$ is a *control vector* or simply *the control*, as well as *input variable* or simply *input*, $\mathbf{y}$ is *an output vector* or simply *output*.

The set of all state vectors (phase vectors) is called *state-space representation* or *a phase space*.

The equation (1.10) is called *the equation of state*, and the equation (1.11) is called *the output equation* or *the observation equation*.

Hereafter, all vectors will be considered as column vectors, i.e.

$$\begin{aligned}\mathbf{x} &= (x_1, x_2, \ldots, x_n)^T;\\ \mathbf{u} &= (u_1, u_2 \ldots, u_r)^T;\\ \mathbf{y} &= (y_1, y_2 \ldots, y_m)^T,\end{aligned} \quad (1.12)$$

where T is a transposition operation, i.e. the replacement of the matrix rows with its columns.

If the input action and the output coordinate of the system are scalar quantities, then such systems are called *one-dimensional* or *systems with one input and one output*. In the English-language literature, such systems are called *single input single output system* (SISO). If at least one of the specified variables is a vector, then such systems are called *multidimensional*. In the English literature, multidimensional systems are divided into:

• *multi input single output system* (MISO), if the input variable is vector;

• *single input multi output system* (SIMO), if the output variable is vector;

• *multi input multi output system* (MIMO), if both input and output variables are vectors.

1.1.3. Converting of equations of linear systems into normal form

In the general case, the equation of a one-dimensional controllable system or object has the form [11]

$$p^n y + a_1 p^{n-1} y + \cdots + a_n y = b_0 p^m u + b_1 p^{m-1} u + \cdots + b_m u,\ m \le n. \quad (1.13)$$

Let us consider three cases separately: $m = 0$, $m = n$, $0 < m < n$.

• $m = 0$. In this case, solving the equations (1.13) for the highest derivative, we obtain

$$p^n y = b_0 u - a_1 p^{n-1} y - \ldots - a_n y. \quad (1.14)$$

Denoting $y = x_1,\ py = x_2, \ldots,\ p^{n-1} y = x_n$, we get

$$\begin{aligned}
&px_1 = x_2;\\
&px_2 = x_3;\\
&\vdots\\
&px_{n-1} = x_n;\\
&px_n = b_0 u - a_n x_1 - a_{n-1} x_2 - a_{n-2} x_3 - \ldots - a_1 x_n,\\
&y = x_1.
\end{aligned} \quad (1.15)$$

In vector form, this system will take the form

$$\begin{aligned}
p\mathbf{x} &= \mathbf{Ax} + \mathbf{Bu};\\
y &= \mathbf{C}^T \mathbf{x},
\end{aligned} \quad (1.16)$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_{n-1} \\ x_n \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-2} & -a_{n-2} & \dots & -a_1 \end{bmatrix};$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ b_0 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix}. \tag{1.17}$$

- $m = n$. In this case, the equations (1.13) can be transformed into the form

$$\begin{aligned} & px_1 = x_2 + k_1 u; \\ & px_2 = x_3 + k_2 u; \\ & \dots \\ & px_{n-1} = x_n + k_{n-1} u; \\ & px_n = -a_n x_1 - a_{n-1} x_2 - a_{n-2} x_3 - \dots - a_1 x_n + k_n u, \\ & y = x_1 + k_0 u, \end{aligned} \tag{1.18}$$

where

$$k_0 = b_0; \quad k_i = \sum_{j=1}^{i} a_j k_{ij}; \quad i = 1,2,\cdots,n.$$

In vector form, this system will take the form

$$\begin{aligned} p\mathbf{x} &= \mathbf{A}\mathbf{x} + \mathbf{B}u; \\ y &= \mathbf{C}^T \mathbf{x} + \mathbf{k_0} u, \end{aligned} \tag{1.19}$$

where

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_{n-1} \\ x_n \end{bmatrix}; \quad \mathbf{A}=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix};$$

$$\mathbf{B}=\begin{bmatrix} k_1 \\ k_2 \\ \dots \\ k_{n-1} \\ k_n \end{bmatrix}; \quad \mathbf{C}=\begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix}. \qquad (1.20)$$

- $0 < m < n$. Let the controllable system be described by the following transfer function

$$W(p)=\frac{y}{u}=\frac{b_0 p^m + b_1 p^{m-1} + \dots + b_m}{a_0 p^n + a_1 p^{n-1} + \dots + a_n}, \quad 0 < m < n. \qquad (1.21)$$

Transforming the transfer function (1.21) as follows

$$y=\frac{b_0 p^m + b_1 p^{m-1} + \dots + b_m}{a_0 p^n + a_1 p^{n-1} + \dots + a_n} u, \quad 0 < m < n, \qquad (1.22)$$

we divide the left and right-hand sides of the resulting expression by its numerator

$$\frac{1}{b_0 p^m + b_1 p^{m-1} + \ldots + b_m} y = \frac{1}{a_0 p^n + a_1 p^{n-1} + \ldots + a_n} u. \qquad (1.23)$$

Let us introduce a new fictitious coordinate x_1, which satisfies the relation

$$x_1 = \frac{1}{b_0 p^m + b_1 p^{m-1} + \ldots + b_m} y = \frac{1}{a_0 p^n + a_1 p^{n-1} + \ldots + a_n} u \qquad (1.24)$$

from which

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$$(a_0 p^n + a_1 p^{n-1} + \cdots + a_n)x_1 = u;$$
$$y = (b_0 p^m + b_1 p^{m-1} + \cdots + b_m)x_1. \qquad (1.25)$$

The first equation of the system (1.25) corresponds to the case $m = 0$, so it can be represented as follows

$$px_1 = x_2;$$
$$px_2 = x_3;$$
$$\vdots$$
$$px_{n-1} = x_n;$$
$$px_n = \frac{1}{a_0}\left(u - a_n x_1 - a_{n-1}x_2 - \ldots - a_1 x_n\right). \qquad (1.26)$$

The second equation of the system (1.25) is the observability equation

$$y = b_0 x_{m+1} + b_1 x_m + \ldots + b_m x_1. \qquad (1.27)$$

Combining the system of equations (1.26) and the observability equation (1.27) , we obtain a system of differential equations of the original object in normal form

$$px_1 = x_2;$$
$$px_2 = x_3;$$
$$\vdots$$
$$px_{n-1} = x_n;$$
$$px_n = \frac{1}{a_0}\left(u - a_n x_1 - a_{n-1}x_2 - \ldots - a_1 x_n\right),$$
$$y = b_0 x_{m+1} + b_1 x_m + \ldots + b_m x_1. \qquad (1.28)$$

1.1.4. Matrix transfer functions

Let us consider a linear time-invariant object

$$p\mathbf{x} = \mathbf{Ax} + \mathbf{Bu}; \quad \mathbf{y} = \mathbf{Cx}, \quad \mathbf{x} \in R^n, \mathbf{y} \in R^r, \mathbf{u} \in R^m. \qquad (1.29)$$

In these equations, n – the vector $\mathbf{x}$ (column matrix $n \times 1$) is called the state vector of the object, r – the vector $\mathbf{y}$, the components of which are the output signals of the object, is called the output vector, and m – the vector $\mathbf{u}$ is the vector of external inputs that are applied to the object from the outside. The object matrix $\mathbf{A}$, control matrix $\mathbf{B}$ and output signal matrix $\mathbf{C}$ have dimensions $n \times n$, $n \times m$ and $r \times n$, respectively.

Let us define the transfer function for the object (1.29) in matrix form. To do this, we move the components containing state variables to the left, leaving only control inputs on the right-hand side. Taking into account that the equations are written in matrix form, after rearrangement we obtain the following equation

$$p\mathbf{Ex} - \mathbf{Ax} = \mathbf{Bu}, \qquad (1.30)$$

or

$$(p\mathbf{E} - \mathbf{A})\mathbf{x} = \mathbf{Bu}. \qquad (1.31)$$

Considering the state variables $\mathbf{x}$ as unknowns, we solve the resulting equations

$$\mathbf{x} = (p\mathbf{E} - \mathbf{A})^{-1}\mathbf{Bu}. \qquad (1.32)$$

Conventionally dividing the left and right-hand sides of the equation (1.32) by $\mathbf{u}$, we obtain the following transfer function

$$\mathbf{W}(p) = \frac{\mathbf{x}(p)}{\mathbf{u}(p)} = (p\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}. \qquad (1.33)$$

Let's define the matrix transfer function of a closed-loop system. In this case, we will assume that all state variables of the object are directly measured and used as object's output signals. Then the matrix $\mathbf{C}$ transforms into the identity matrix $\mathbf{E}$, so the second equation of the system (1.29) can be written as

$$\mathbf{y} = \mathbf{x}. \qquad (1.34)$$

The controller, which controls the object, receives the object's state variables as input signals $x_1, \ldots, x_n$ and generates control inputs that are applied to the object. We will assume that the controller is linear, meaning that the generated control inputs are a linear combination of input signals.

The output signals of the controller can be applied to the object at the same points through which the measured external inputs are applied.

Denoting these external inputs through $\mathbf{v}$, and $(m \times n)$ – controller transformation matrix through $\mathbf{P}$, we get total control applied to the object

$$\mathbf{u} = \mathbf{v} - \mathbf{Px}. \tag{1.35}$$

The minus sign in front of the second term indicates the negative feedback on the state variables of the control object.

By combining equations (1.29) and (1.35), we obtain the following closed-loop system equation:

$$p\mathbf{x} = \mathbf{Ax} + \mathbf{Bu} = \mathbf{Ax} + \mathbf{B(v - Px)} = \mathbf{(A - BP)x} + \mathbf{Bv}. \tag{1.36}$$

From the equation (1.36), the matrix transfer function of the closed-loop system can be obtained [34]

$$\mathbf{\Phi}(p) = \frac{\mathbf{x}(p)}{\mathbf{v}(p)} = (p\mathbf{E} - \mathbf{A} + \mathbf{BP})^{-1}\mathbf{B}. \tag{1.37}$$

1.2. Development of mathematical models of electromechanical systems

1.2.1. Fundamental laws of electromechanics

The operation of any electric machine can be described with considerable accuracy by four fundamental physical laws.

• Kirchhoff's (Ohm) law

$$U = I \cdot R + E_{ci} + E_{\omega}, \tag{1.38}$$

where U is a voltage applied to the winding, I, R are current through this winding and its resistance, E_{ci} is EMF od self-induction, caused by the change in magnetic flux generated by the winding, E_{ω} is rotational EMF, associated with the interaction of magnetic fluxes from different windings.

This law indicates the distribution of voltage applied to the winding of an electric machine.

• Faraday's law

$$E = -\frac{d\Phi}{dt}, \tag{1.39}$$

Where Φ — magnetic flux in a machine.

This law explains the causes and conditions for the occurrence of EMF in the winding of an electric machine.

• Ampere's law

$$dF = I \cdot [\overline{dl}, \overline{B}], \tag{1.40}$$

where I is a current through an infinitesimally conductor, $\overline{dl}$ is a length of an infinitesimally conductor, $\overline{B}$ is inductance of the magnetic field created by current I .

Ampere's law determines the force acting on a current-carrying conductor in a magnetic field and connects electromagnetic and mechanical processes in an electric machine.

• Newton's law for rotational motion

$$J\frac{d\omega}{dt} = \sum_{i=1}^{n} M_i, \tag{1.41}$$

where J, ω, M_i are moment of inertia, angular velocity and torques acting on the motor shaft.

Newton's law indicates the acceleration with which the motor shaft will move and the torques that cause this movement.

Depending on the type of electric machine, the above laws are transformed accordingly [16] and form systems of equations that describe a particular electric machine.

For example, for a DC machine with series excitation, the equations (1.38)—(1.41) will take the form

$$U = IR + L\frac{dI}{dt} + Kk_{\Phi}I\omega;$$
$$M_e = Kk_{\Phi}I^2;$$
$$J\frac{d\omega}{dt} = M_e - M_c, \tag{1.42}$$

where U is a supply voltage, I – current in the armature circuit, ω – shaft rotation speed, R, L – resistance and inductance of the armature circuit, k_{Φ} – linearization coefficient of the motor magnetization curve, M_c – load torque on its shaft, J – moment of motor inertia, K – design coefficient.

In the case of studying the "Motor - Industrial Mechanism" system, where elastic deformations in the kinematic chains cannot be neglected, multi-mass mechanical systems are considered. In this case, the equations (1.42) or similar ones must be supplemented with an equation based on Hooke's law [8]

$$dM_y = cd\theta, \tag{1.43}$$

Where c, θ are elasticity coefficient and angle of twist of the shaft, respectively.

Considering that the motor shaft twists because its ends have different speeds ω_1 and ω_2, that is,

$$\frac{d\theta}{dt} = \omega_1 - \omega_2 \tag{1.44}$$

the equation (1.43) can be represented as follows

$$\frac{dM_y}{dt} = c(\omega_1 - \omega_2). \tag{1.45}$$

Then the motion of a two-mass mechanical system is described by the equations

$$J_1\frac{d\omega_1}{dt} = M_e - M_y;$$
$$\frac{dM_y}{dt} = c(\omega_1 - \omega_2);$$
$$J_2\frac{d\omega_2}{dt} = M_y - M_c, \tag{1.46}$$

where J_1, J_2 are moments of inertia of the motor and working mechanism.

If there is elasticity not only between the motor and the working mechanism, but also between the individual parts of the industrial mechanism, the system (1.46) is written for all significant masses.

From a physical point of view, the system of equations (1.46) is conservative because it does not take into account energy dissipation during elastic deformations. Dissipative mechanical systems where energy dissipation occurs during deformations, can be described if the equation (1.45) is supplemented by a component that takes into account dissipation

$$\frac{dM_y}{dt} = c \cdot (\omega_1 - \omega_2) + \beta \cdot \left(\frac{d\omega_1}{dt} - \frac{d\omega_2}{dt} \right), \qquad (1.47)$$

where β is the energy dissipation coefficient during elastic deformations.

When the electromechanical system operates in positioning or trajectory-following modes, the equations of its dynamics must be supplemented by the dependence

$$p\varphi = \omega. \qquad (1.48)$$

In addition to the elasticity of the mechanical transmission, the dynamics of the electromechanical system (EMS) is significantly influenced by its operating conditions. Thus, when the working body moves in viscous media, a friction torque arises, which is generally described by the equation:

$$M_c = h \cdot \omega^{\alpha}, \qquad (1.49)$$

where h is the viscosity of the working medium, α – an empirical exponent that takes into account the mechanical characteristics of the production mechanism.

Thus, the behavior of any EMS can be described, using the above equations

However, in cases where not only the control system of an electric drive, but also the control system of the technological process or the industrial mechanism as a whole is being studied, it is necessary to consider the dynamics of this process or mechanism taking into account known physical laws.

To formulate the corresponding equations, a clear understanding of the laws of theoretical mechanics, fluid dynamics, thermal engineering, chemistry, etc. is necessary. In most cases, these laws are formulated mathematically as partial differential equations, but with a certain accuracy they can be reduced to linear differential equations.

For example, the process of heating the windings of an electric machine is described by the heat conduction equation

$$c\rho \cdot \frac{dT}{dt} = \lambda \cdot \left(\frac{d^2T}{dx^2} + \frac{d^2T}{dy^2} + \frac{d^2T}{dz^2} \right), \qquad (1.50)$$

where c, ρ, λ are the heat capacity, density and thermal conductivity of the winding material, respectively, T is the winding temperature.

To solve equation (1.50) correctly, the winding design must be specified, and the initial and boundary conditions must be defined. But even in this case, solving the equation (1.50) is quite complex and requires special numerical methods for solving partial differential equations.

However, a sufficiently accurate temperature distribution can be obtained by solving the approximate heat balance equation [8]

$$Q \cdot dt = A \cdot \tau dt + C \cdot d\tau, \qquad (1.51)$$

where Q, A, C are the amount of heat generated in the machine, heat dissipation from the electric machine to the environment and the machine's heat capacity.

This equation is based on the assumption of homogeneity of materials and design of the electrical machine.

After some simple transformations, the heat balance equation (1.51)can be presented in the form of an ordinary linear differential equation

$$\frac{Q}{A} = \tau + \frac{C}{A} \cdot \frac{d\tau}{dt}. \qquad (1.52)$$

Thus, in the future, to describe any processes occurring in electromechanical systems, we will use ordinary differential equations.

To simplify the notation, we will agree to use the operator form of writing

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differential equations and will use the following designations

$$\frac{df(x)}{dt} \to pf(x); \quad \int f(x)dx \to \frac{f(x)}{p}. \qquad (1.53)$$

Taking into account the designations (1.53), the equation (1.52) in operator form takes the form

$$\frac{Q}{A} = \tau + \frac{C}{A} p\tau. \qquad (1.54)$$

In the following discussion we will deal with object whose differential equations of motion are solved with respect to the highest derivative, so we transform the equation (1.54) as follows

$$p\tau = -\frac{A}{C} \cdot \tau + \frac{Q}{C}. \qquad (1.55)$$

The equation (1.55) is a differential equation in normal or Cauchy form. Not only single equations, but also systems of them can be written in this form.

Substituting the value of the electromagnetic torque from the second equation of the system (1.42) into the third equation of the same system and introducing the electromagnetic time constant into consideration

$$T_e = \frac{L}{R}, \qquad (1.56)$$

we can write the system (1.42), supplemented with equations (1.48) and (1.49) in normal form

$$\begin{aligned} p\varphi &= \omega; \\ p\omega &= \frac{Kk_\Phi}{J} I^2 - \frac{h}{J}\omega^\alpha; \\ pI &= -\frac{1}{T_e} I - \frac{Kk_\Phi}{RT_e}\omega I + \frac{1}{RT_e} U. \end{aligned} \qquad (1.57)$$

Hereinafter, the angle of rotation φ, speed ω and current I of the motor will be called the coordinates of the control object, voltage U — controlling input, or control.

By generalizing equations (1.55) and (1.57), we can conclude that the dynamics of any electromechanical object is determined by three components:

• Coordinates, which will include the EMF of the converter, currents in the windings, torques, speeds, rotation angles, etc. We will denote the coordinates of the EMF as X_i, considering i as the index of the variable.

The numbering of variables is carried out from the output of the system to its input. That is, in a speed control system of a DC motor, the first variable is the speed , the second is the armature current and the third is the EMF of the converter.

• Control inputs U_k, which are control voltages generated by controls. The index k corresponds to the number of the controlled variable. Thus, the control that is formed by the speed control U_1, current control U_2, etc.

• Disturbances F_j acting on the control object. The index j corresponds to the number of the equation in which this disturbance appears. For example, the load torque M_c is included in the first equation of the system (1.57) so it is denoted F_2.

Taking into account the above, the motion of a generalized electromechanical object can be represented as follows:

$$pX_i = \sum_{j=1}^{n} b_{ij} \cdot X_j + m_i \cdot U_i + c_i \cdot F_i, \quad i = 1,2,\ldots,n, \tag{1.58}$$

where b_{ij}, m_i, c_i are the coefficients for the object coordinates, control inputs and disturbances, respectively.

1.2.2. Normalization of differential equations of the control object

The material presented in the previous section reflects the essence of the physical processes occurring in the control object, but it significantly complicates the analysis and solution of the dynamic equations. The difficulties arise due to the use of named physical quantities.

For example, when analyzing the dynamic or static properties of an electric drive, which is described by equations (1.57), a situation may arise when it becomes necessary to sum the second and third equations. From a physical point of view, such an action is unacceptable, since the equations differ in the dimensions of physical quantities – the second equation has the dimension rad/s, and the third – A/s.

In order to eliminate this inconsistency and enable performing any operations on the dynamic equations regardless of the units of measurement, use *the operation of directed normalization* is used.

This operation involves switching to a new phase space, which differs from the original space in scale along each of the axes of the coordinate basis. In this case, the axes are scaled by switching to relative units. Mathematically, this can be represented as follows

$$x_i = \frac{X_i}{X_{\mathrm{bi}}}, \qquad (1.59)$$

where X_i, X_{bi} are the current and base value of i-th coordinate of the control object.

In different fields of knowledge, the choice of a base quantity is controversial.

We will assume that for electric motor with series excitation, the rated current, nominal speed, nominal rotation angle, and nominal voltage are used as the base quantities.

If we now divide the current, speed and position of the motor shaft by the corresponding base quantities, we obtain their relative values

$$y_1 = \frac{\varphi}{\varphi_{\mathrm{b}}}; y_2 = \frac{\omega}{\omega_{\mathrm{b}}}; y_3 = \frac{I}{I_{\mathrm{b}}}. \qquad (1.60)$$

The coefficients of normalized differential equations can be determined through the parameters of the control object and the accepted base quantities.

Using the accepted relative units, we will make the transition from equations in named quantities (1.57) to motion equations in relative units. These equations are not accidentally solved with respect to the derivative – it is the left-hand side of the

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equations that form the system (1.57)that determines which base quantity should be used to divide a given equation.

Let's consider the first equation of the system (1.57).

In order to keep this equation unchanged, we divide its left and right-hand sides by the base value – the nominal rotation angle φ_b. Let us assume that the nominal rotation angle is the angle through which the motor rotates at nominal speed ω_b in one second, i.e.

$$\varphi_b = \omega_b \cdot 1c. \tag{1.61}$$

Taking this notation into account, the first equation of the system (1.57) in relative units will take the form

$$p\frac{\varphi}{\varphi_b} = \frac{\omega}{\varphi_b} = \frac{\omega}{\omega_b \cdot 1c} \tag{1.62}$$

or

$$py_1 = y_2. \tag{1.63}$$

Now let's consider the second equation of system (1.57). We divide its left and right parts by the nominal speed ω_b

$$p\frac{\omega}{\omega_b} = \frac{Kk_\Phi}{J\omega_b}I^2 - \frac{h}{J\omega_b}\omega^\alpha . \tag{1.64}$$

The resulting equation shows the change in the relative value of the motor speed, but it still contains named units on the right-hand side – armature current I and motor speed ω.

We eliminate these units by multiplying and dividing each component by the corresponding base quantity

$$p\frac{\omega}{\omega_b} = \frac{Kk_\Phi}{J\omega_b}I^2\left(\frac{I_b}{I_b}\right)^2 - \frac{h}{J\omega_b}\omega^\alpha\left(\frac{\omega_b}{\omega_b}\right)^\alpha \tag{1.65}$$

or

$$p\frac{\omega}{\omega_b}=\frac{Kk_\Phi}{J\omega_b}I_b^2\left(\frac{I}{I_b}\right)^2-\frac{h}{J\omega_b}\omega_b^\alpha\left(\frac{\omega}{\omega_b}\right)^\alpha. \tag{1.66}$$

Taking relative units (1.60) into account, the equations (1.66) can be written as follows

$$py_2=\frac{Kk_\Phi I_b^2}{J\omega_b}y_3^2-\frac{h\omega_b^{\alpha-1}}{J}y_2^\alpha. \tag{1.67}$$

The resulting equation (1.67) relates the relative values of acceleration py_2, speed y_2 and motor current y_3, i.e. is the equation of motion in relative units.

We will denote the coefficients of the equations of motion in Cauchy form and relative units which are associated with the coordinates of the object by a Latin letter b with two indices: the first indicating the number of the variable in the left-hand side of the equation, and the second indicating the number of the variable with which this coefficient is associated. Then, taking into account the notation

$$b_{22}=-\frac{h\omega_b^{\alpha-1}}{J};\quad b_{23}=\frac{Kk_\Phi I_b^2}{J\omega_b}, \tag{1.68}$$

the equation (1.67) takes the form

$$py_2=b_{22}y_2^\alpha+b_{23}y_3^2. \tag{1.69}$$

We divide the left and right-hand sides of the third equation of the system (1.57) by the rated current

$$p\frac{I}{I_b}=-\frac{1}{T_e}\cdot\frac{I}{I_b}-\frac{Kk_\Phi}{R\cdot T_e}\cdot\frac{I}{I_b}\cdot\omega+\frac{1}{R\cdot T_e}\cdot\frac{1}{I_b}\cdot U \tag{1.70}$$

or

$$p\frac{I}{I_b}=-\frac{1}{T_e}\cdot\frac{I}{I_b}-\frac{Kk_\Phi}{R\cdot T_e}\cdot\frac{I}{I_b}\cdot\frac{\omega}{\omega_b}\cdot\omega+\frac{1}{R\cdot T_e}\cdot\frac{1}{I_b}\cdot\frac{U}{U_b}U_b. \tag{1.71}$$

Using the relations (1.60), we substitute the relative value of the armature current into the equation (1.70)

$$py_3 = -\frac{1}{T_e} \cdot y_3 - \frac{Kk_\Phi}{R \cdot T_e} \cdot \omega_b \cdot y_3 \cdot y_2 + \frac{1}{R \cdot T_e} \cdot \frac{1}{I_b} \cdot U_b U_3, \tag{1.72}$$

where $U_3 = \frac{U}{U_b}$ is the supply voltage in relative units.

The equation (1.72) shows the change in the relative value of the armature current depending on the relative values of speed and voltage.

For the convenience of further use of the resulting equation, we introduce the coefficients

$$m_3 = \frac{1}{R \cdot T_e} \cdot \frac{1}{I_b} \cdot U_b; \quad b_{32} = -\frac{k_\Phi}{R \cdot T_e} \cdot \omega_b; \quad b_{33} = -\frac{1}{T_b}. \tag{1.73}$$

and represent it as follows

$$py_3 = b_{32} y_2 y_3 + b_{33} y_3 + m_3 U_3. \tag{1.74}$$

Analysis of the equation (1.74) with coefficients (1.73) indicates that with dimensionless relative coordinates of the object, the dimension of this equation is determined only by the dimension of the coefficients (1.73). Transition to relative units made it possible to write down the motion equations of the electromechanical object (1.57), which have different dimensions, in the form of equations in relative units

$$py_1 = b_{12} y_2; \; py_2 = b_{22} y_2^\alpha + b_{23} y_3^2;$$
$$py_3 = b_{32} y_2 y_3 + b_{33} y_3 + m_3 U_3, \tag{1.75}$$

where

$$b_{12} = 1. \tag{1.76}$$

The dimension of each equation in the system (1.75) is c^{-1}. The corresponding block diagram of the control object is shown in Fig. 1.1.

The same rule applies to motion equations in relative units of any dynamic objects.

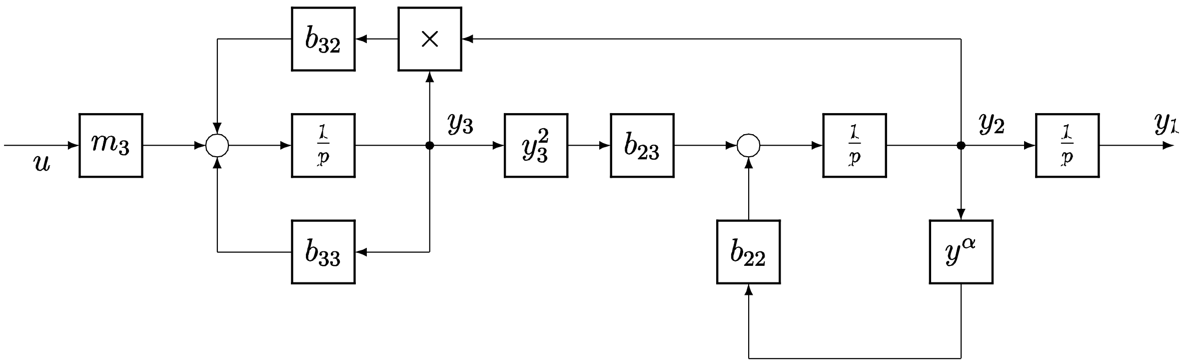

Fig. 1.1. Block diagram of the control object

Previously, the principles of compiling matrix equations for the dynamics of linear electromechanical objects were considered. The dynamics of nonlinear objects can also be described by differential equations in matrix form. However, unlike the linear case, all matrices will be column matrices of dimension $n \times 1$. Further we will call such matrices n-matrices.

Let us demonstrate the compilation of dynamic equations for nonlinear objects in matrix form using the example of equations (1.75). The standard matrix form of writing the motion equations of a nonlinear object has the following form

$$p\mathbf{y} = \mathbf{f}(\mathbf{y}) + \mathbf{g}(\mathbf{y})U, \tag{1.77}$$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \ \mathbf{f}(\mathbf{y}) = \begin{pmatrix} y_2 \\ b_{22}y_2^\alpha + b_{23}y_3^2 \\ b_{32}y_2y_3 + b_{33}y_3 \end{pmatrix}, \ \mathbf{g}(\mathbf{y}) = \begin{pmatrix} 0 \\ 0 \\ m_3 \end{pmatrix}. \tag{1.78}$$

The composition of matrices $\mathbf{y}$ and $\mathbf{g}(\mathbf{y})$ is similar to the linear case, and the matrix $\mathbf{f}(\mathbf{y})$ contains the right-hand sides of the corresponding equations, excluding control and perturbing influences.

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1.3. Controllability and stabilization of dynamic objects

1.3.1. Concept of controllability of dynamic objects

One of the fundamental concepts in automatic control theory is controllability. To clarify the general laws and patterns to which controllable systems are subject, let us consider a dynamic system, whose motion is described by the matrix equation

$$p\mathbf{x} = \mathbf{f}(\mathbf{x},\mathbf{u},t), \quad x \in R^n, u \in R^r, \tag{1.79}$$

where $\mathbf{x}$ is the n-dimensional state vector, $\mathbf{u}$ is the r-dimensional control vector.

A control input $\mathbf{u} = \mathbf{u}(t) = (u_1(t)\, u_2(t), \ldots, u_r(t))^T$ is *piecewise continuous* if all its components $u_i(t)$ are piecewise continuous. Piecewise continuous controls are called *admissible*.

The system (1.79) is called *controllable* or *fully controllable* if for any initial deviations $\mathbf{x}_0$ in the phase space R^n on a finite interval $[t_0, t_k]$ it is possible to determine an admissible control that transfers the system (1.79) from the initial point $\mathbf{x}(t_0) = \mathbf{x}_0$ to the final one $\mathbf{x}(t_k) = \mathbf{x}_k$.

In other words, if an object is fully controllable, then it can be transferred by admissible control from an arbitrary initial state to any other state within a finite time.

1.3.2. Controllability of linear objects

For a linear dynamic object, which can be described by the equation

$$p\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \ \mathbf{x} \in R^n, \mathbf{u} \in R^r, \tag{1.80}$$

the following statement is true:

• the linear object (1.80) is fully controllable if, regardless of the initial position $\mathbf{x}(t_0) = \mathbf{x}_0$, there is an admissible control, defined on a finite interval $[t_0, t_k]$, that transfers the object (1.80) to the final state $\mathbf{x}(t_k) = \mathbf{0}$, i.e. to the origin.

• the linear object (1.80) is fully controllable if, regardless of the final state $\mathbf{x}(t_k) = \mathbf{x}^k$, there is an admissible control, defined on a finite interval $[t_0, t_k]$, that transfers the object (1.80) from the initial state $\mathbf{x}(t_0) = \mathbf{0}$, that is, from the origin, to the final state $\mathbf{x}(t_k) = \mathbf{x}^k$.

The statement remains valid if in the first part the final point $\mathbf{x}(t_k) = \mathbf{0}$ is replaced with any other fixed point, and in the second part the initial point $\mathbf{x}(t_0) = \mathbf{0}$ is replaced with any other fixed point.

1.3.3. Controllability of linear time-invariant objects

Let the equation

$$p\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x} \in R^n, \mathbf{u} \in R^r. \tag{1.81}$$

describe a time-invariant system, that is, matrices $\mathbf{A}$ and $\mathbf{B}$ contain constant coefficients.

Let us consider the matrix

$$\mathbf{Y} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}. \tag{1.82}$$

This matrix is called *the controllability matrix*.

Controllability criterion for linear time-invariant systems. *The linear time-invariant object is fully controllable if and only if the controllability matrix has a full rank, i.e. when rank equals* n.

Recall that the rank of a matrix is equal to the number of independent rows, the number of independent columns, and the order of the non-zero minor of the maximum dimension.

A pair $(\mathbf{A}, \mathbf{B})$ is called *controllable* or *fully controllable* if the rank of the controllability matrix (1.82) equals n.

As an example, we will show that an electric motor with a transfer function

$$\frac{\omega}{u}=\frac{K}{T_e T_{\mathrm{m}} p^2+T_{\mathrm{m}} p+1}, \tag{1.83}$$

where K is the design coefficient of the motor,

is fully controllable.

To demonstrate this, let's compose the equations of motion of the object under study, by representing its transfer function as follows:

$$\frac{\omega}{K}=\frac{u}{T_{\mathrm{e}} T_{\mathrm{m}} p^2+T_{\mathrm{m}} p+1}=x_1, \tag{1.84}$$

from which we obtain the equations

$$y=Kx_1;\, u=T_{\mathrm{e}} T_{\mathrm{m}} p^2 x_1+T_{\mathrm{m}} p x_1+x_1. \tag{1.85}$$

The second equation can be replaced by two first-order equations describing the motion of the motor

$$px_1=x_2; \quad px_2=\frac{1}{T_{\mathrm{e}} T_{\mathrm{m}}}\left(-x_1-T_{\mathrm{m}} x_2+u\right). \tag{1.86}$$

The first equation of the system (1.86) allows us to write the following observability equation

$$\omega=Kx_1. \tag{1.87}$$

From equations (1.86) and (1.87) the object parameter matrices are determined

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -1 & -T_{\mathrm{m}}\end{bmatrix};\, \mathbf{B}=\begin{bmatrix}0\\ 1\end{bmatrix};\, \mathbf{C}=\begin{bmatrix}0\\ K\end{bmatrix}. \tag{1.88}$$

Before proceeding to compiling the controllability matrix $\mathbf{Y}$, we calculate the product $\mathbf{AB}$

$$\mathbf{AB}=\begin{bmatrix}0 & 1\\ -1 & -T_{\mathrm{m}}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}1\\ -T_{\mathrm{m}}\end{bmatrix}. \tag{1.89}$$

Let’s create a controllability matrix for the object under consideration

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$$\mathbf{Y} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -T_m \end{bmatrix}. \tag{1.90}$$

The rank of the matrix $\mathbf{Y}$ equals 2, which is confirmed by the presence of non-zero determinant and minors of the matrix $\mathbf{Y}$.

1.3.4. Invariance of the controllability property of linear transformations

Consider the non-singular linear transformation

$$\mathbf{x} = \mathbf{Tz};\ \det(\mathbf{T}) \neq 0. \tag{1.91}$$

Under this transformation, the equations (1.80) take the form

$$p\mathbf{z} = \mathbf{Az} + \mathbf{B}u, \tag{1.92}$$

where

$$\mathbf{A} = \mathbf{T}^{-1}\mathbf{AT};\ \mathbf{B} = \mathbf{T}^{-1}\mathbf{BT}. \tag{1.93}$$

Let's verify that the pair $(\mathbf{A},\mathbf{B})$ is fully controllable. Let's create a controllability matrix for this pair

$$\mathbf{Y} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B}. \end{bmatrix} \tag{1.94}$$

It is easy to verify that

$$\mathbf{A}^k = \mathbf{T}^{-1}\mathbf{A}^k\mathbf{T},\ \mathbf{B}^k = \mathbf{T}^{-1}\mathbf{B}^k\mathbf{T}. \tag{1.95}$$

Substituting expressions (1.95) into matrix (1.94) , we obtain

$$\mathbf{Y} = \mathbf{T}^{-1}\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} = \mathbf{T}^{-1}\mathbf{Y}. \tag{1.96}$$

Since the matrix $\mathbf{T}^{-1}$ is non-singular and its rank equals n, than the rank of the controllability matrix $\mathbf{Y}$ of the transformed object equals to the rank of the controllability matrix $\mathbf{Y}$ of the original object.

The controllability property remains unchanged under a non-singular linear transformation of matrices.

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1.3.5. Controllability subspace

The region consisting of all points of the state space to which a controllable system can be transferred by admissible control from the origin within a finite time is called its *controllability region*.

If a controllable system is fully controllable, then its controllability region coincides with the entire space. If the rank of the controllability matrix of the controllable system is not equal to the maximum value (i.e., the dimension of the state space), but is greater than zero, then the controllable system is said to be *not fully controllable* or *partially controllable*. If the controllable system is partially controllable, then, as follows from the controllability criterion, the controllability region coincides with the subspace generated by the set of independent columns of the controllability matrix. This subspace is called *the controllability subspace*.

1.3.6. Canonical form of controllability

Let the rank of the controllability matrix of a linear time-invariant controllable system (1.80) be equal to $l\ (l \le n)$. Let us consider a non-singular transformation $\mathbf{z} = \mathbf{T}\mathbf{x}$, where the transformation matrix has the form $\mathbf{T} = [\mathbf{T_1}, \mathbf{T_2}]$ and is constructed as follows: $\mathbf{T}_1$ is $(n \times l)$-matrix, and its columns are l independent columns of the controllability matrix; $\mathbf{T_2}$ is $(n \times (n-l))$-matrix, and its columns are chosen so that the matrix $\mathbf{T}$ is non-singular. Under this transformation, the equation (1.80) takes the form of the so-called *canonical form of controllability*

$$\begin{bmatrix} p\mathbf{z}^{(1)} \\ p\mathbf{z}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ 0 & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{z}^{(1)} \\ \mathbf{z}^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{B_1} \\ 0 \end{bmatrix} u, \tag{1.97}$$

or

$$p\mathbf{z}^{(1)} = \mathbf{A_{11}}\mathbf{z}^{(1)} + \mathbf{A_{12}}\mathbf{z}^{(2)} + \mathbf{B_1}u;$$
$$p\mathbf{z}^{(2)} = \mathbf{A_{22}}\mathbf{z}^{(2)}, \qquad (1.98)$$

where $\mathbf{z}^{(1)}$ is the l-vector, $\mathbf{z}^{(2)}$ – $(n-l)$-vector, $\mathbf{A}_{11}$ – $(l \times l)$-matrix, $\mathbf{A}_{12}$ – $(l \times nl)$-matrix, $\mathbf{A}_{22}$ – $(nl \times nl)$-matrix, $\mathbf{B}_1$ – $(l \times r)$-matrix.

From the structure of the equations (1.98) it is clear that the vector $\mathbf{z}^{(2)}$ is uncontrollable, as its change is not influenced by the control either directly or through other phase coordinates. The vector $\mathbf{z}^{(1)}$ is fully controllable, meaning that it can be adjusted as needed by selecting the appropriate control.

As an example, let us transform the equation

$$px_1 = x_2 + x_3 + u;$$
$$px_2 = x_2;$$
$$px_3 = x_1;$$
$$px_4 = -x_2 - x_4. \qquad (1.99)$$

into the canonical form of controllability.

The matrices $\mathbf{A},\ \mathbf{B}$ and their products $\mathbf{AB},\ \mathbf{A}^2\mathbf{B},\ \mathbf{A}^3\mathbf{B}$ have the following form:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$
$$\mathbf{AB} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad \mathbf{A}^2\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{A}^3\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \qquad (1.100)$$

Composing the controllability matrix from these matrices, we obtain

$$\mathbf{Y} = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B}\end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (1.101)$$

The controllability matrix has two independent columns, so its rank is 2.

We choose matrices $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}$ as follows:

$$\mathbf{T_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix};\ \mathbf{T_2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix};\ \mathbf{T} = \begin{bmatrix}\mathbf{T_1} & \mathbf{T_2}\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (1.102)$$

The transformation $\mathbf{z} = \mathbf{Tx}$ has the form

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix}. \qquad (1.103)$$

The equations in the new variables take the form

$$\begin{aligned} pz_1 &= px_1 = x_2 + x_3 + u = z_3 + z_2 + u; \\ pz_2 &= px_3 = x_1 = z_1; \\ pz_3 &= px_2 = x_2 = z_3; \\ pz_4 &= px_4 = -x_2 - x_4 = -z_3 - z_4, \end{aligned} \qquad (1.104)$$

or

$$\begin{aligned} pz_1 &= z_2 + z_3 + u; \\ pz_2 &= z_1; \\ pz_3 &= z_3; \\ pz_4 &= -z_3 - z_4. \end{aligned} \qquad (1.105)$$

Using vector notation $\mathbf{z}^{(1)} = (z_1 \quad z_2)^T$ and $\mathbf{z}^{(2)} = (z_3 \quad z_4)^T$, these equations can be written as

$$p\mathbf{z}^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{z}^{(1)} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{z}^{(2)} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u;$$
$$p\mathbf{z}^{(2)} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}\mathbf{z}^{(2)}. \qquad (1.106)$$

Analysis of the equations (1.106) shows that the phase coordinate z_1 is directly controlled by the corresponding input; the phase coordinate z_2 is affected by control through z_1 (z_1 is included in the equation z_2), but the phase coordinates z_3 and z_4 are not affected by control, that is, these coordinates are uncontrollable.

Thus, the use of a non-singular transformation made it possible to group controllable z_1, z_2 and uncontrollable z_3, z_4 state variables.

1.3.7. Canonical forms of equations

Considering that there are many equivalent forms of equations of state, one can always choose the most convenient one for use in a particular case. Such forms of equations are called *canonical*. Their variety is explained by the diversity of tasks in which they are used.

Let us consider transformations of the state equations into the canonical form, called *the Luenberger controllable form*

$$p\mathbf{z} = \mathbf{A}\mathbf{z} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}u. \qquad (1.107)$$

The characteristic equation of the matrix À of this equation has the form

$$\det(p\mathbf{E}-\mathbf{A}) = p^n + a_1 p^{n-1} + \ldots + a_n = 0. \qquad (1.108)$$

The coefficients of the characteristic equation are the elements of the last row of the matrix $\mathbf{A}$ in equation (1.107) taken with the opposite sign.

In order to make a non-singular transformation of the state equation

$$p\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}u,\ \mathbf{x} \in R^n,\ u \in R, \qquad (1.109)$$

and turn it into a Luenberger controllable form (1.107), it is necessary and sufficient for the pair $(\mathbf{A}, \mathbf{B})$ to be fully controllable.

Under a non-singular transformation, the characteristic equations of the original system (1.108) and the transformed system (1.107) $\det(p\mathbf{E}-\mathbf{A})=0$ and $\det(p\mathbf{E}-\mathbf{A})=0$ are identical.

As an example, let us transform the state equation

$$p\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}u \qquad (1.110)$$

into Luenberger's controllable form. The products $\mathbf{AB}$, $\mathbf{A}^2\mathbf{B}$ and the controllability matrix $\mathbf{Y}$ have the form

$$\mathbf{AB} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix};\ \mathbf{A}^2\mathbf{B} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix};$$

$$\mathbf{Y} = \left[\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}\right] = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}. \qquad (1.111)$$

Since $\det(\mathbf{Y})=1$, the pair $(\mathbf{A}, \mathbf{B})$ is fully controllable. Therefore, the considered equation can be transformed into a Luenberger controllable form.

The characteristic equation has the form

$$\det\left(p\mathbf{E}-\mathbf{A}\right)=\begin{bmatrix} p+1 & 1 & 0 \\ 0 & p+1 & 0 \\ -1 & 0 & p+1 \end{bmatrix}=$$

$$=\left(p+1\right)^3=p^3+3p^2+3p+1=0. \tag{1.112}$$

Thus, the elements of the last row of the matrix $\mathbf{A}$ will be $a_1=-3; a_2=-3;$ $a_3=-1$, and the transformed equation takes the form

$$p\mathbf{z}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}\mathbf{z}+\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u \quad . \tag{1.113}$$

1.3.8. Invariance of characteristic equations roots of systems to linear transformations

The roots of the characteristic equation of linear time-invariant systems do not change under a linear non-singular transformation. Let us prove this statement by considering a linear time-invariant system, described by the equation

$$p\mathbf{x}=\mathbf{A}\mathbf{x}+\mathbf{B}u,\ \mathbf{x}\in R^n, u\in R^r, \tag{1.114}$$

Under a non-special transformation $\mathbf{z}=\mathbf{T}\mathbf{x}$, this equation is transformed to the form

$$p\mathbf{z}=\hat{\mathbf{A}}\mathbf{z}+\hat{\mathbf{B}}u, \tag{1.115}$$

where $\hat{\mathbf{A}}=\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$, $\hat{\mathbf{B}}=\mathbf{T}^{-1}\mathbf{B}$.

The characteristic equation of the original system has the form

$$\det(p\mathbf{E}-\mathbf{A})=|p\mathbf{E}-\mathbf{A}|=0, \tag{1.116}$$

and the characteristic equation of the transformed system has following form

$$\det(p\mathbf{E}-\hat{\mathbf{A}})=|p\mathbf{E}-\hat{\mathbf{A}}|=0. \tag{1.117}$$

Let us show that these equations have the same roots.

By multiplying the matrix $p\mathbf{E}-\mathbf{A}$ on the left by $\mathbf{T}$, and on the right by $\mathbf{T}^{-1}$, we get

$$\mathbf{T}(p\mathbf{E}-\mathbf{A})\mathbf{T}^{-1} = p\mathbf{E}-\hat{\mathbf{A}}. \tag{1.118}$$

Thus, taking into account that the determinant of the product of square matrices is equal to the product of the determinants of their factors for the characteristic equation of the transformed system, we find

$$|\mathbf{T}||(p\mathbf{E}-\mathbf{A})||\mathbf{T}^{-1}| = |p\mathbf{E}-\hat{\mathbf{A}}| = 0. \tag{1.119}$$

Since the matrix $\mathbf{T}$ is non-singular, then $|\mathbf{T}|\neq 0; |\mathbf{T}^{-1}|\neq 0$ therefore

$$|(p\mathbf{E}-\mathbf{A})| = |p\mathbf{E}-\hat{\mathbf{A}}| = 0, \tag{1.120}$$

that proves the invariance of characteristic equations of linear systems, the dynamics of which are represented in different phase spaces. In return, identical characteristic equations cannot have different roots. Thus, the invariance of the roots is proven.

1.3.9. Stabilization of linear time-invariant systems

One of the important concepts when defining control problems is *stabilization*. A controllable system is called stabilizable if there is a control law under which the closed-loop system is asymptotically stable.

Linear time-invariant object

$$p\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \ \mathbf{x}\in R^n, u\in R^r, \tag{1.121}$$

is called stabilizable if there is a control law $u=\mathbf{K}\mathbf{x}$ under which the closed-loop system $p\mathbf{x} = (\mathbf{A}+\mathbf{B}\mathbf{K})\mathbf{x}$ is asymptotically stable.

If an object is fully controllable, then it is stabilizable. However, the opposite is not true: an object may be stabilizable, but not fully controllable. Therefore, the problem of determining the stabilization criterion arises.

1.3.10. Stabilization criterion

Let us consider a non-singular transformation $\mathbf{x} = \mathbf{Tz}$ that transforms the equation of the controllable system into the canonical form:

$$p\mathbf{z}^{(1)} = \mathbf{A}_{11}\mathbf{z}^{(1)} + \mathbf{A}_{12}\mathbf{z}^{(2)} + \mathbf{B}_1 u;$$
$$p\mathbf{z}^{(2)} = \mathbf{A}_{22}\mathbf{z}^{(2)}. \quad (1.122)$$

The stabilizability of system described by these equations means that there is a control law $u = \mathbf{Kz}$ under which the vector variables $\mathbf{z}^{(1)}$ and $\mathbf{z}^{(2)}$ of the closed-loop system tend to zero at $t \to \infty$. Since in the first of the above equations the pair $(\mathbf{A}_{11}, \mathbf{B}_{11})$ is fully controllable, then there is a control law $u = \mathbf{Kz}^{(1)}$ under which the closed-loop subsystem described by this equation is stable, and the vector variable $\mathbf{z}^{(1)}$ will tend to zero if the vector variable $\mathbf{z}^{(2)}$ (which in this equation acts as an external disturbance) will also tend to zero at $t \to \infty$. But, as follows from the second equation, the latter will take place if and only if the matrix $\mathbf{A}_{22}$ in the second equation is stable, i.e. the roots of its characteristic equation have negative real parts.

Stabilization criterion: for a linear time-invariant controllable system, which is not fully controllable, to be stabilizable, it is necessary and sufficient that the matrix $\mathbf{A}_{22}$ in the canonical form of controllability be stable.

Let us study the stabilizability of the controllable system, which is described by the equations

$$px_1 = x_2 + x_3 + u; \quad px_2 = x_2;$$
$$px_3 = x_1; \quad px_4 = -x_2 - x_4 \quad . \quad (1.123)$$

This system was considered in the previous example and, as it was proven, these equations in the canonical controllability form take the form

$$p\mathbf{z}^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{z}^{(1)} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{z}^{(2)} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u;$$

$$p\mathbf{z}^{(2)} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}\mathbf{z}^{(2)}. \qquad (1.124)$$

In order for the system under consideration to be stabilizable, according to the Stabilization criterion, the matrix $\mathbf{A}_{22}$ must be stable. The characteristic equation of this matrix has the form

$$\det\begin{pmatrix} p-1 & 0 \\ 1 & p+1 \end{pmatrix} = p^2 - 1 = 0 . \qquad (1.125)$$

The roots of this equation are $p_{1,2} = \pm 1$, i.e. the given matrix is unstable, and, consequently, the system is not stabilizable.

1.4. Feedback linearization

1.4.1. Concept of affine and non-affine dynamic systems

Regardless of the form of representation of the equations of motion of an electromechanical system of n-th order, its dynamics can be conveniently studied by examining the phase trajectories represented in a space whose basis is defined by the state variables of the system under consideration and the independent variable – time. This approach is because the projections of the motion trajectory onto the corresponding coordinate planes determine both the phase portraits of the system and its transient processes. Furthermore, the geometric interpretation of the processes occurring in the system makes it possible to use the tools of analytical and differential geometry to analyze the features of the movement of this system.

One of the basic concepts of analytical geometry, which has found application in control theory, is the concept of affine transformation, which led to the emergence of the concept of affine system [17].

Let us consider this concept in more detail.

An affine transformation (from the Latin *affinis* – connected with, close, adjacent) is a mapping of a plane or space in which straight lines are mapped to straight lines. The affine transformation is defined by the following expression

$$f(\mathbf{x}) = \mathbf{M}\mathbf{x} + \mathbf{v}, \tag{1.126}$$

where $\mathbf{x}$ is the vector of system state variables, $\mathbf{M}$ is the invertible transformation matrix, $\mathbf{v}$ is the vector of coordinate displacements.

In other words, a transformation is called affine if it can be obtained as follows:

• Choose a new space basis with a new origin $\mathbf{v}$;

• Assign a point $f(x)$ that has the same coordinates relative to the new coordinate system as x in the old one for each point x in space.

Affine transformations are transformations of motion and similarity. There are two types of transformations:

• *Equiaffine transformation* is an affine transformation that preserves area (affine length is also preserved).

• *Centroaffine transformation* is an affine transformation that preserves the origin.

An example of an affine transformation is the transformation of one triangle into another (Fig. 1.2).

This transformation is defined by the following expression

$$(x, y) \mapsto (y - 100, 2 \cdot x + y - 100), \tag{1.127}$$

Since a change in coordinates of a stationary electromechanical system occurs under the influence of external inputs, we will refer to systems as affine ones if they are linear with respect to these inputs. Among affine systems, we will distinguish those that are affine in control, the dynamics of which are described by differential equations of the form

$$p\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})\mathbf{u}, \tag{1.128}$$

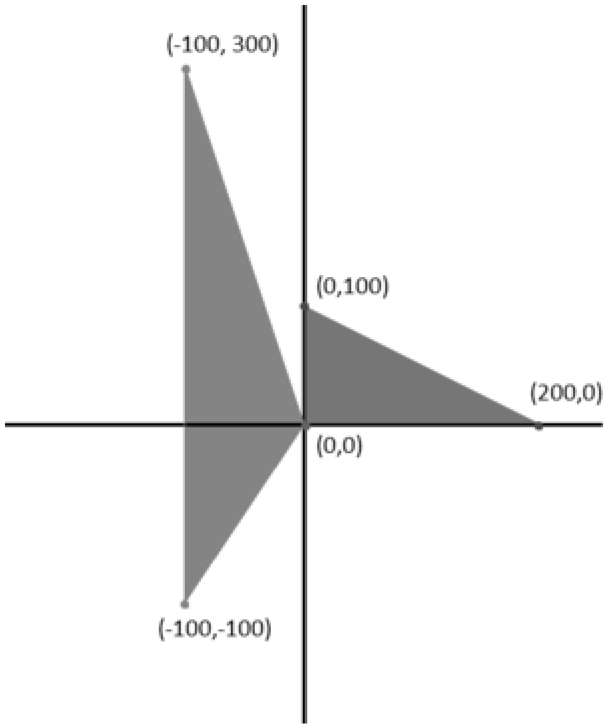

Fig. 1.2. Affine transformation: motion and similarity

where $\boldsymbol{\eta}$ is a vector of state variables, $\mathbf{f}(\boldsymbol{\eta}), \mathbf{g}(\boldsymbol{\eta})$ are some functions, $\mathbf{u}$ is a vector of control inputs.

If the vector of derivative state variables of the electromechanical system depends nonlinearly on the vector of control inputs,

$$p\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta}, \mathbf{u}), \tag{1.129}$$

then such systems will be called non-affine.

In the general case, the presence of nonlinear functions $\mathbf{f}(\boldsymbol{\eta}), \mathbf{g}(\boldsymbol{\eta}), \mathbf{g}(\boldsymbol{\eta}, \mathbf{u})$ in the mathematical description of electromechanical systems significantly complicates the analysis of their characteristics and the synthesis of desired motion trajectories. Even in the case when functions $\mathbf{f}(\boldsymbol{\eta}), \mathbf{g}(\boldsymbol{\eta}), \mathbf{g}(\boldsymbol{\eta}, \mathbf{u})$ degenerate into linear dependencies, the nature of the processes occurring in the system differs significantly with variations in the roots of its characteristic equation and their distribution, caused by changes in the parameters of the system. Therefore, it is convenient to study the characteristics and processes occurring in the system, as well as the synthesis of the corresponding control inputs for a normalized system, followed by taking into account the features of the system under study.

We will consider a dynamical system of n-th order as a normalized system

$$px_1 = x_2; \quad px_2 = x_3; \quad \cdots \quad px_n = v. \tag{1.130}$$

or in matrix form

$$p\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{v}, \tag{1.131}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}; \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}; \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{1.132}$$

with n-fold zero root of the characteristic equation

$$p^n x_1 = 0. \tag{1.133}$$

The system (1.130) is affine and can be obtained from systems (1.128) and (1.129) using feedback transformations [11].

1.4.2. Feedback linearization principle

Let us consider that the nonlinear function $\mathbf{\Phi}(\mathbf{x}, \mathbf{v})$ is known and used in the formation of the nonlinear control input

$$\mathbf{u} = \mathbf{\Phi}(\mathbf{x}, \mathbf{v}). \tag{1.134}$$

An equation (1.134) is called *a feedback transformation* if a vector of new controls $\mathbf{v}$ can be determined from it.

According to [11], feedback linearization is the reduction of the motion equations of a known nonlinear object

$$p\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u \tag{1.135}$$

to a controlled Brunovsky form using feedback transformation (1.134)

$$p\eta_1 = \eta_2; \quad p\eta_2 = \eta_3; \quad \cdots \quad p\eta_{n-1} = \eta_n; \quad p\eta_n = v. \tag{1.136}$$

Feedback linearization (FL) is not an approximate, but an equivalent transformation, so the use of FL allows to obtain a system equivalent to the original one. In FL, control $\mathbf{u}$ is replaced by new control $\mathbf{v}$. The transformation function includes not only the new control but also the state vector (in a particular case, only one original variable), i.e. the object is encompassed by feedback in such a transformation. Hence the name – feedback transformation.

Let us consider FL for an electromechanical object whose dynamics equations have the form

$$p\eta = -a\eta^2 + bu. \tag{1.137}$$

We will look for a control input that makes the closed-loop system (1.137) asymptotically stable, and the motion equation (1.137) takes the form

$$p\eta = \nu. \tag{1.138}$$

To do this, we accept the control input as

$$u = \frac{1}{b}\left(a\eta^2 + \nu\right) = \frac{a}{b}\eta^2 + \frac{1}{b}\nu. \tag{1.139}$$

Substituting the found control into the equation (1.137), we obtain the desired motion equation

$$p\eta = -a\eta^2 + b\left[\frac{a}{b}\eta^2 + \frac{1}{b}\nu\right] = -a\eta^2 + a\eta^2 + \nu = \nu. \tag{1.140}$$

Thus, the control input (1.139) allows to compensate for the nonlinear component $-a\eta^2$ and normalize the intensity of the reference signal, bringing the coefficient b to unity. Using the control input on an object (1.137)

$$u = \frac{1}{b}\left(a\eta^2 + \nu\right) = \frac{a}{b}\eta^2 - \frac{g}{b}\eta \tag{1.141}$$

brings the equation of motion of a closed-loop system to the form

$$p\eta = -g\eta. \tag{1.142}$$

Analysis of the obtained equation indicates that using a transformation (1.139), the dynamics of a closed-loop control system for a nonlinear object is described by a linear equation corresponding to the motion equation of a first-order aperiodic element.

Thus, the given example allows us to formulate the principle of feedback linearization: the control input applied to the object must contain two components: the first compensates for the nonlinearity of the control object through coordinate transformation, and the second ensures the desired movement of the linearized object. Moreover, the motion trajectories of the object are completely determined by the parameters of the selected controller, that forms the control input ν. It should be noted that, unlike other methods, feedback linearization allows to reduce not only nonlinear but also linear objects to the Brunovsky form. For the latter, feedback transformation compensates for the internal feedbacks of the object, its time constants and gain factors.

Let us confirm this statement with the following example: let it be necessary to present the motion equation of a generalized first-order electromechanical object in Brunovsky form. Let us write the motion equation of the studied object in deviations

$$p\eta = a_{11}\eta + m_1 u. \qquad (1.143)$$

By equating the derivative $p\eta$ to the new control input ν

$$\nu = p\eta, \qquad (1.144)$$

we define a control input u that compensates for internal feedback $a_{11}\eta$ and gain factor m_1

$$u = \frac{1}{m_1}\left(\nu - a_{11}\eta\right). \qquad (1.145)$$

Substituting the control input (1.145) into the equation (1.143) allows to present it in Brunovsky form

$$\begin{aligned} p\eta &= a_{11}\eta + m_1\frac{1}{m_1}\left(\nu - a_{11}\eta\right) = \\ &= a_{11}\eta + \frac{m_1}{m_1}\nu - m_1\frac{a_{11}\eta}{m_1} = \\ &= a_{11}\eta + \nu - a_{11}\eta = \nu. \end{aligned} \qquad (1.146)$$

The considered example concerns the case when nonlinearity and control are linearly related. However, this is only a particular case of nonlinear systems. In general, determining the linearizing control input, which brings the motion equations to the Brunovsky form, is carried out using the methods of differential geometry and Lie group theory.

1.4.3. Introduction to Differential Geometry and Lie Group Theory

When further considering issues of feedback linearization, the following concepts must be defined: derivatives of a function with respect to a vector argument, Lie derivatives and brackets, diffeomorphisms, involutivity, and integrability of a system of linearly independent vectors.

Let us consider a function $\alpha(\boldsymbol{\eta})$ that is a smooth function of a vector variable $\boldsymbol{\eta} = (\eta_1 \quad \dots \quad \eta_n)$ and a vector function $\mathbf{f}(\boldsymbol{\eta}) = (f_1(\eta_1 \quad \dots \quad \eta_n) \quad \dots \quad f_n(\eta_1 \quad \dots \quad \eta_n)$, whose components depend on the components of the vector variable $\boldsymbol{\eta}$.

To denote the operation of differentiation of these functions we will use the operator

$$\nabla = \frac{d}{d\boldsymbol{\eta}} = \begin{pmatrix} \dfrac{\partial}{\partial \eta_1} & \cdots & \dfrac{\partial}{\partial \eta_n} \end{pmatrix}. \tag{1.147}$$

When this operator is applied to a scalar function $\alpha(\boldsymbol{\eta})$, we obtain a row vector

$$\nabla \alpha = \frac{d}{d\boldsymbol{\eta}} \alpha(\boldsymbol{\eta}) = \begin{pmatrix} \dfrac{\partial \alpha(\boldsymbol{\eta})}{\partial \eta_1} & \cdots & \dfrac{\partial \alpha(\boldsymbol{\eta})}{\partial \eta_n} \end{pmatrix}, \tag{1.148}$$

and when differentiating the vector function $\mathbf{f}(\boldsymbol{\eta})$ we obtain the matrix

$$\nabla\mathbf{f}(\boldsymbol{\eta}) = \frac{d}{d\boldsymbol{\eta}}\mathbf{f}(\boldsymbol{\eta}) = \begin{pmatrix} \frac{\partial f_1(\boldsymbol{\eta})}{\partial\eta_1} & \cdots & \frac{\partial f_1(\boldsymbol{\eta})}{\partial\eta_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(\boldsymbol{\eta})}{\partial\eta_1} & \cdots & \frac{\partial f_n(\boldsymbol{\eta})}{\partial\eta_n} \end{pmatrix}. \quad (1.149)$$

As an example, let us differentiate the scalar $\alpha(\boldsymbol{\eta}) = a_1\eta_1^2 + a_2\eta_1\eta_2$ and vector $\mathbf{f}(\boldsymbol{\eta}) = \begin{pmatrix} a_{11}\eta_1^2 + a_{12}\eta_1\eta_2 \\ a_{21}\eta_1\eta_2 + a_{22}\eta_2^2 \end{pmatrix}$ functions of a vector variable $\boldsymbol{\eta} = (\eta_1 \quad \eta_2)$

$$\nabla\alpha = \frac{d}{d\boldsymbol{\eta}}\alpha(\boldsymbol{\eta})\begin{pmatrix} \frac{\partial\alpha(\boldsymbol{\eta})}{\partial\eta_1} & \frac{\partial\alpha(\boldsymbol{\eta})}{\partial\eta_2} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\partial(a_1\eta_1^2 + a_2\eta_1\eta_2)}{\partial\eta_1} & \frac{\partial(a_1\eta_1^2 + a_2\eta_1\eta_2)}{\partial\eta_2} \end{pmatrix} =$$

$$= \begin{pmatrix} 2a_1\eta_1 + a_2\eta_2 & a_2\eta_1 \end{pmatrix} \quad (1.150)$$

and

$$\nabla\mathbf{f}(\boldsymbol{\eta}) = \frac{d}{d\boldsymbol{\eta}}\mathbf{f}(\boldsymbol{\eta}) = \begin{pmatrix} 2a_{11}\eta_1 & a_{12}\eta_2 \\ a_{21}\eta_2 & a_{21}\eta_1 + 2a_{22}\eta_2 \end{pmatrix}. \quad (1.151)$$

The Lie derivative of a scalar function $\alpha(\boldsymbol{\eta})$ with respect to a vector function $\mathbf{f}(\boldsymbol{\eta})$ is the scalar function $L_f\alpha$ that is defined by following ratio

$$L_f\alpha = \frac{d\alpha(\boldsymbol{\eta})}{d\boldsymbol{\eta}}\mathbf{f}(\boldsymbol{\eta}) = \nabla\alpha(\boldsymbol{\eta})\mathbf{f}(\boldsymbol{\eta}) = \sum_{i=1}^{n}\frac{\partial\alpha(\boldsymbol{\eta})}{\partial\eta_i}f_i(\boldsymbol{\eta}). \quad (1.152)$$

The high order Lie derivatives are determined recursively

$$L_f^k\alpha = L_f\left(L_f^{k-1}\alpha\right) = \nabla\left(L_f^{k-1}\alpha\right)\mathbf{f}. \quad (1.153)$$

The zero derivative of a Lie function $\alpha(\boldsymbol{\eta})$ with respect to $\mathbf{f}(\boldsymbol{\eta})$ is the function itself $\alpha(\boldsymbol{\eta})$

$$L_f^0 \alpha = \alpha. \tag{1.154}$$

The Lie derivatives considered are equivalent to the first and highest partial derivatives $\frac{\partial f(\eta)}{\partial \eta}$ and $\frac{\partial^n f(\eta)}{\partial \eta^n}$ from classical differential calculus. Similarly, in differential geometry there are Lie derivatives, which are analogues of derivatives of the form $\frac{\partial^2 f(\eta,\mu)}{\partial \eta \partial \mu}$. Thus, the Lie derivative of a function $\alpha(\boldsymbol{\eta})$ with respect to functions $\mathbf{f}(\boldsymbol{\eta})$ and $\mathbf{g}(\boldsymbol{\eta})$ is defined as follows

$$\begin{aligned} L_g L_f \alpha &= L_g \left(L_f \alpha \right) = \\ &= \nabla \left(L_f \alpha \right) \mathbf{g}(\boldsymbol{\eta}) = \nabla \left(\nabla \alpha(\boldsymbol{\eta}) \mathbf{f}(\boldsymbol{\eta}) \right) \mathbf{g}(\boldsymbol{\eta}) = \\ &= \sum_{i=1}^{n} \frac{\partial \sum_{i=1}^{n} \frac{\partial \alpha(\boldsymbol{\eta})}{\partial \eta_i} f_i(\boldsymbol{\eta})}{\partial \eta_i} g_i(\boldsymbol{\eta}). \end{aligned} \tag{1.155}$$

As an example, consider the perturbed motion of a nonlinear EMS in state space

$$p\eta_1 = a_{12}\eta_2^2; \quad p\eta_2 = a_{21}\eta_2 + a_{22}\eta_1\eta_2 + m_2 U. \tag{1.156}$$

We will assume that the measured coordinate of the system is related to the state variables by the following observability equation

$$y(\eta_1, \eta_2) = c_1\eta_1\eta_2 + c_2\eta_2. \tag{1.157}$$

Let us represent the system (1.156) in matrix form

$$p\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})U, \tag{1.158}$$

where

$$\mathbf{f}(\boldsymbol{\eta}) = \begin{pmatrix} f_1(\boldsymbol{\eta}) \\ f_2(\boldsymbol{\eta}) \end{pmatrix} = \begin{pmatrix} a_{12}\eta_2^2 \\ a_{21}\eta_2 + a_{22}\eta_1\eta_2 \end{pmatrix};$$

$$\mathbf{g}(\boldsymbol{\eta}) = \begin{pmatrix} g_1(\boldsymbol{\eta}) \\ g_2(\boldsymbol{\eta}) \end{pmatrix} = \begin{pmatrix} 0 \\ m_2 \end{pmatrix} \tag{1.159}$$

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and find the first derivatives of the Lie function $y(\eta_1, \eta_2)$ with respect to the functions $\mathbf{f}(\boldsymbol{\eta})$ and $\mathbf{g}(\boldsymbol{\eta})$.

According to the equation (1.152) the derivative $L_f y$ will be

$$L_f y = \sum_{i=1}^{2} \frac{\partial y(\boldsymbol{\eta})}{\partial \eta_i} f_i(\boldsymbol{\eta}) = \frac{\partial y(\boldsymbol{\eta})}{\partial \eta_1} f_1(\boldsymbol{\eta}) + \frac{\partial y(\boldsymbol{\eta})}{\partial \eta_2} f_2(\boldsymbol{\eta}) =$$

$$= \frac{\partial (c_1 \eta_1 \eta_2 + c_2 \eta_2)}{\partial \eta_1} (a_{12} \eta_2^2) +$$

$$+ \frac{\partial (c_1 \eta_1 \eta_2 + c_2 \eta_2)}{\partial \eta_2} (a_{21} \eta_2 + a_{22} \eta_1 \eta_2) =$$

$$= c_1 \eta_2 a_{12} \eta_2^2 + (c_1 \eta_1 + c_2)(a_{21} \eta_2 + a_{22} \eta_1 \eta_2) =$$

$$= c_1 a_{12} \eta_2^3 + c_1 a_{21} \eta_1 \eta_2 + c_2 a_{21} \eta_2 + c_1 a_{22} \eta_1^2 \eta_2 + c_2 a_{22} \eta_1 \eta_2 . \quad (1.160)$$

Similarly, the Lie derivative with respect to the function $g(\boldsymbol{\eta})$ is determined

$$L_g y = \sum_{i=1}^{2} \frac{\partial y(\boldsymbol{\eta})}{\partial \eta_i} g_i(\boldsymbol{\eta}) = \frac{\partial y(\boldsymbol{\eta})}{\partial \eta_1} g_1(\boldsymbol{\eta}) + \frac{\partial y(\boldsymbol{\eta})}{\partial \eta_2} g_2(\boldsymbol{\eta}) =$$

$$= \frac{\partial (c_1 \eta_1 \eta_2 + c_2 \eta_2)}{\partial \eta_1} 0 + \frac{\partial (c_1 \eta_1 \eta_2 + c_2 \eta_2)}{\partial \eta_2} m_2 =$$

$$= c_1 m_2 \eta_1 + c_2 m_2 . \quad (1.161)$$

Using the dependencies (1.153) and (1.155), we determine the second derivatives $L_f^2 y, L_g^2 y, L_g L_f y, L_f L_g y$.

$$L_f^2 y = L_f (L_f y) = \nabla (L_f y) \mathbf{f}(\boldsymbol{\eta}) = \frac{\partial L_f y}{\partial \eta_1} f_1(\boldsymbol{\eta}) + \frac{\partial L_f y}{\partial \eta_2} f_2(\boldsymbol{\eta}) =$$

$$= (c_1 a_{21} \eta_2 + c_2 a_{21} \eta_2 + 2 c_1 a_{22} \eta_1 \eta_2 + c_2 a_{22} \eta_2) a_{12} \eta_2^2 +$$

$$+ (3 c_1 a_{12} \eta_2^2 + c_1 a_{21} \eta_1 + c_2 a_{21} + c_1 a_{22} \eta_1^2 + c_2 a_{22} \eta_1)(a_{21} \eta_1 \eta_2 + a_{22} \eta_2);$$

(1.162)

$$L_g^2 y = L_g(L_g y) = \nabla(L_g y)\mathbf{g}(\boldsymbol{\eta}) = \frac{\partial L_g y}{\partial \eta_1} g_1(\boldsymbol{\eta}) + \frac{\partial L_g y}{\partial \eta_2} g_2(\boldsymbol{\eta}) =$$
$$= c_1 m_2 \times 0 + 0 \times m_2 = 0; \qquad (1.163)$$

$$L_g L_f y = L_g(L_f y) = \nabla(L_f y)\mathbf{g}(\boldsymbol{\eta}) = \frac{\partial L_f y}{\partial \eta_1} g_1(\boldsymbol{\eta}) + \frac{\partial L_f y}{\partial \eta_2} g_2(\boldsymbol{\eta}) =$$
$$= (c_1 a_{21}\eta_2 + c_2 a_{21}\eta_2 + 2c_1 a_{22}\eta_1\eta_2 + c_2 a_{22}\eta_2) \times 0 +$$
$$+ (3c_1 a_{12}\eta_2^2 + c_1 a_{21}\eta_1 + c_2 a_{21} + c_1 a_{22}\eta_1^2 + c_2 a_{22}\eta_1) m_2; \qquad (1.164)$$

$$L_f L_g y = L_f(L_g y) = \nabla(L_g y)\mathbf{f}(\boldsymbol{\eta}) = \frac{\partial L_g y}{\partial \eta_1} f_1(\boldsymbol{\eta}) + \frac{\partial L_g y}{\partial \eta_2} f_2(\boldsymbol{\eta}) =$$
$$= c_1 m_2 a_{12}\eta_2^2 + 0 \times (a_{21}\eta_1\eta_2 + a_{22}\eta_2). \qquad (1.165)$$

Now let us move on to the differentiation of vector functions and consider two smooth vector functions $\mathbf{f}(\boldsymbol{\eta})$ and $\mathbf{g}(\boldsymbol{\eta})$. The definition of Lie derivatives in this case is similar to the scalar case, but differs from it in that the result of differentiating one vector function with respect to another is the vector

$$L_f \mathbf{g} = \frac{d\mathbf{g}(\boldsymbol{\eta})}{d\boldsymbol{\eta}} \mathbf{f}(\boldsymbol{\eta}) = \nabla \mathbf{g}(\boldsymbol{\eta}) \mathbf{f}(\boldsymbol{\eta}) =$$
$$= \left(\sum_{i=1}^{n} \frac{\partial g_1(\boldsymbol{\eta})}{\partial \eta_i} f_i(\boldsymbol{\eta}) \;\; \sum_{i=1}^{n} \frac{\partial g_2(\boldsymbol{\eta})}{\partial \eta_i} f_i(\boldsymbol{\eta}) \;\; \vdots \;\; \sum_{i=1}^{n} \frac{\partial g_n(\boldsymbol{\eta})}{\partial \eta_i} f_i(\boldsymbol{\eta}) \right)^T. \qquad (1.166)$$

Using the values of the functions $\mathbf{f}(\boldsymbol{\eta})$ and $\mathbf{g}(\boldsymbol{\eta})$ accepted in the previous example, we determine the derivatives $L_f \mathbf{g}$ and $L_g \mathbf{f}$.

$$L_f\mathbf{g}=\nabla\mathbf{g}(\boldsymbol{\eta})\mathbf{f}(\boldsymbol{\eta})\begin{pmatrix}\frac{\partial g_1(\boldsymbol{\eta})}{\partial\eta_1} & \frac{\partial g_1(\boldsymbol{\eta})}{\partial\eta_2}\\ \frac{\partial g_2(\boldsymbol{\eta})}{\partial\eta_1} & \frac{\partial g_2(\boldsymbol{\eta})}{\partial\eta_2}\end{pmatrix}\begin{pmatrix}f_1(\boldsymbol{\eta})\\ f_2(\boldsymbol{\eta})\end{pmatrix}=$$

$$=\begin{pmatrix}\frac{\partial g_1(\boldsymbol{\eta})}{\partial\eta_1}f_1(\boldsymbol{\eta})+\frac{\partial g_1(\boldsymbol{\eta})}{\partial\eta_2}f_2(\boldsymbol{\eta})\\ \frac{\partial g_2(\boldsymbol{\eta})}{\partial\eta_1}f_1(\boldsymbol{\eta})+\frac{\partial g_2(\boldsymbol{\eta})}{\partial\eta_2}f_2(\boldsymbol{\eta})\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}; \tag{1.167}$$

$$L_g\mathbf{f}=\nabla\mathbf{f}(\boldsymbol{\eta})\mathbf{g}(\boldsymbol{\eta})=\begin{pmatrix}\frac{\partial f_1(\boldsymbol{\eta})}{\partial\eta_1} & \frac{\partial f_1(\boldsymbol{\eta})}{\partial\eta_2}\\ \frac{\partial f_2(\boldsymbol{\eta})}{\partial\eta_1} & \frac{\partial f_2(\boldsymbol{\eta})}{\partial\eta_2}\end{pmatrix}\begin{pmatrix}g_1(\boldsymbol{\eta})\\ g_2(\boldsymbol{\eta})\end{pmatrix}=$$

$$=\begin{pmatrix}\frac{\partial f_1(\boldsymbol{\eta})}{\partial\eta_1}g_1(\boldsymbol{\eta})+\frac{\partial f_1(\boldsymbol{\eta})}{\partial\eta_2}g_2(\boldsymbol{\eta})\\ \frac{\partial f_2(\boldsymbol{\eta})}{\partial\eta_1}g_1(\boldsymbol{\eta})+\frac{\partial f_2(\boldsymbol{\eta})}{\partial\eta_2}g_2(\boldsymbol{\eta})\end{pmatrix}=$$

$$=\begin{pmatrix}0\times0+2a_{12}m_2\eta_2\\ a_{21}\eta_2\times0+(a_{21}\eta_1+a_{22})m_2\end{pmatrix}=$$

$$=\begin{pmatrix}2a_{12}m_2\eta_2\\ (a_{21}\eta_1+a_{22})m_2\end{pmatrix}. \tag{1.168}$$

Analysis of the expression (1.167) indicates that the derivative $L_f\mathbf{g}$ for dynamic systems with constant coefficients under control inputs is a matrix-vector with zero elements.

Lie brackets of functions $\mathbf{f}(\boldsymbol{\eta})$ and $\mathbf{g}(\boldsymbol{\eta})$ are a vector function defined by the ratio

$$[f,g]=ad_f g=\nabla\mathbf{gf}-\nabla\mathbf{fg}=L_\mathbf{f}\mathbf{g}-L_\mathbf{g}\mathbf{f}. \tag{1.169}$$

Using the dependence (1.169), we write first-order Lie brackets for the previously defined functions $\mathbf{f}(\boldsymbol{\eta})$ and $\mathbf{g}(\boldsymbol{\eta})$.

$$ad_f g = L_{\mathbf{f}}\mathbf{g} - L_{\mathbf{g}}\mathbf{f} = \begin{pmatrix} (ad_f g)_1 \\ (ad_f g)_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2a_{12}m_2\eta_2 \\ (a_{21}\eta_1 + a_{22})m_2 \end{pmatrix} = -\begin{pmatrix} 2a_{12}m_2\eta_2 \\ a_{21}m_2\eta_1 + a_{22}m_2 \end{pmatrix}. \quad (1.170)$$

High order Lie brackets are defined recursively

$$ad_f^k g = ad_f\left(ad_f^{k-1} g\right) = \left[\mathbf{f}, ad_f^{k-1} g\right] \quad (1.171)$$

For vector functions $\mathbf{f}$ and $\mathbf{g}$, which were considered in previous examples, we find the value of the second-order Lie bracket

$$ad_f^2 g = ad_f\left(ad_f g\right) = \nabla(ad_f g)\mathbf{f} - \nabla\mathbf{f} ad_f g =$$

$$= \begin{pmatrix} \dfrac{\partial(ad_f g)_1}{\partial\eta_1} & \dfrac{\partial(ad_f g)_1}{\partial\eta_2} \\ \dfrac{\partial(ad_f g)_2}{\partial\eta_1} & \dfrac{\partial(ad_f g)_2}{\partial\eta_2} \end{pmatrix} \begin{pmatrix} f_1(\boldsymbol{\eta}) \\ f_2(\boldsymbol{\eta}) \end{pmatrix} -$$

$$- \begin{pmatrix} \dfrac{\partial f_1(\boldsymbol{\eta})}{\partial\eta_1} & \dfrac{\partial f_1(\boldsymbol{\eta})}{\partial\eta_2} \\ \dfrac{\partial f_2(\boldsymbol{\eta})}{\partial\eta_1} & \dfrac{\partial f_2(\boldsymbol{\eta})}{\partial\eta_2} \end{pmatrix} \begin{pmatrix} (ad_f g)_1 \\ (ad_f g)_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 2a_{12}m_2 \\ a_{21}m_2 & 0 \end{pmatrix} \begin{pmatrix} a_{12}\eta_2^2 \\ a_{21}\eta_1\eta_2 + a_{22}\eta_2 \end{pmatrix} -$$

$$- \begin{pmatrix} 0 & 2a_{12}\eta_2 \\ a_{21}\eta_2 & a_{21}\eta_1 + a_{22} \end{pmatrix} \left(-\begin{pmatrix} 2a_{12}m_2\eta_2 \\ a_{21}m_2\eta_1 + a_{22}m_2 \end{pmatrix} \right). \quad (1.172)$$

The zero-order Lie brackets of functions $\mathbf{f}(\boldsymbol{\eta})$ and $\mathbf{g}(\boldsymbol{\eta})$ are identical to the function $\mathbf{g}(\boldsymbol{\eta})$

$$ad_f^0 g = \mathbf{g}. \quad (1.173)$$

Lie brackets are characterized by the following properties [11]:

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• bilinearity

$$[\alpha_1\mathbf{f_1} + \alpha_2\mathbf{f_2}, \mathbf{g}] = \alpha_1[\mathbf{f_1}, \mathbf{g}] + \alpha_2[\mathbf{f_2}, \mathbf{g}]; \qquad (1.174)$$

• asymmetric commutativity

$$[\mathbf{f}, \mathbf{g}] = -[\mathbf{g}, \mathbf{f}]; \qquad (1.175)$$

• Jacobi identity

$$L_{ad_{\mathbf{f}}\mathbf{g}}\alpha = L_{\mathbf{f}}L_{\mathbf{g}}\alpha - L_{\mathbf{g}}L_{\mathbf{f}}\alpha. \qquad (1.176)$$

A smooth vector function $\boldsymbol{\mu} = \boldsymbol{\Phi}(\boldsymbol{\eta})$ defined in a domain Ω is *a diffeomorphism* if there is a uniquely defined smooth inverse function $\boldsymbol{\eta} = \boldsymbol{\Phi}^{-1}(\boldsymbol{\mu})$.

If the functions $\boldsymbol{\Phi}(\boldsymbol{\eta})$ and $\boldsymbol{\Phi}^{-1}(\boldsymbol{\mu})$ are both smooth and defined over the entire space of real numbers, then $\boldsymbol{\Phi}(\boldsymbol{\eta})$ it is a global diffeomorphism.

As an example, let us determine whether the vector function is a diffeomorphism

$$\boldsymbol{\Phi}(\boldsymbol{\eta}) = \begin{pmatrix} f_1(\boldsymbol{\eta}) \\ f_2(\boldsymbol{\eta}) \end{pmatrix} = \begin{pmatrix} \eta_1 + \eta_2^2 \\ \eta_2 \end{pmatrix}.$$

Let us find the Jacobian of the function $\boldsymbol{\Phi}(\boldsymbol{\eta})$

$$\frac{d\boldsymbol{\Phi}(\boldsymbol{\eta})}{d\boldsymbol{\eta}} = \begin{pmatrix} \frac{\partial f_1(\boldsymbol{\eta})}{\partial \eta_1} & \frac{\partial f_1(\boldsymbol{\eta})}{\partial \eta_2} \\ \frac{\partial f_2(\boldsymbol{\eta})}{\partial \eta_1} & \frac{\partial f_2(\boldsymbol{\eta})}{\partial \eta_2} \end{pmatrix} = \begin{pmatrix} 1 & 2\eta_2 \\ 0 & 1 \end{pmatrix}. \qquad (1.177)$$

Analysis of the obtained expression indicates non-zero values of the Jacobian rank for any coordinates of the electromechanical system. A function $\boldsymbol{\Phi}(\boldsymbol{\eta})$ that has this property is called a global diffeomorphism.

Now let us determine whether a vector function $\hat{\boldsymbol{\Phi}}(\boldsymbol{\eta}) = \begin{pmatrix} f_1(\boldsymbol{\eta}) \\ f_2(\boldsymbol{\eta}) \end{pmatrix} = \begin{pmatrix} \eta_2^2 \\ \eta_1\eta_2 + \eta_2 \end{pmatrix}$ is a global diffeomorphism.

Let us find the Jacobian of the function $\hat{\boldsymbol{\Phi}}(\boldsymbol{\eta})$

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$$\frac{d\hat{\mathbf{\Phi}}(\mathbf{\eta})}{d\mathbf{\eta}} = \begin{pmatrix} \frac{\partial f_1(\mathbf{\eta})}{\partial \eta_1} & \frac{\partial f_1(\mathbf{\eta})}{\partial \eta_2} \\ \frac{\partial f_2(\mathbf{\eta})}{\partial \eta_1} & \frac{\partial f_2(\mathbf{\eta})}{\partial \eta_2} \end{pmatrix} = \begin{pmatrix} 0 & 2\eta_2 \\ \eta_2 & 1 \end{pmatrix}. \tag{1.178}$$

Analysis of the obtained expression shows that the rank of the Jacobian takes zero values at the origin. Such function $\hat{\mathbf{\Phi}}(\mathbf{\eta})$ is a local diffeomorphisms.

Diffeomorphisms are used to transform nonlinear systems. Let us consider such a transformation for a nonlinear system

$$p\mathbf{\eta} = \mathbf{f}(\mathbf{\eta}) + \mathbf{g}(\mathbf{\eta})\mathbf{u}. \tag{1.179}$$

We will assume that there is a diffeomorphism $\mathbf{\mu} = \mathbf{\Phi}(\mathbf{\eta})$, then, considering that

$$p\mathbf{\mu} = \frac{\partial \mathbf{\Phi}(\mathbf{\eta})}{\partial \mathbf{\eta}} p\mathbf{\eta} = \frac{\partial \mathbf{\Phi}(\mathbf{\eta})}{\partial \mathbf{\eta}} \left(\mathbf{f}(\mathbf{\eta}) + \mathbf{g}(\mathbf{\eta})\mathbf{u}\right), \tag{1.180}$$

we get

$$p\mathbf{\mu} = \widetilde{\mathbf{f}}(\mathbf{\mu}) + \widetilde{\mathbf{g}}(\mathbf{\mu})\mathbf{u}, \tag{1.181}$$

where $\widetilde{\mathbf{f}}(\mathbf{\mu}) = \left.\frac{\partial \mathbf{\Phi}(\mathbf{\eta})}{\partial \mathbf{\eta}} \mathbf{f}(\mathbf{\eta})\right|_{\mathbf{\eta}=\mathbf{\Phi}^{-1}(\mathbf{\mu})}$ and $\tilde{\mathbf{g}}(\mathbf{\mu}) = \left.\frac{\partial \mathbf{\Phi}(\mathbf{\eta})}{\partial \mathbf{\eta}} \mathbf{g}(\mathbf{\eta})\right|_{\mathbf{\eta}=\mathbf{\Phi}^{-1}(\mathbf{\mu})}$.

A function $\mathbf{\mu} = \mathbf{\Phi}(\mathbf{\eta})$, defined in a vicinity of Ω, is a diffeomorphism in this vicinity if the Jacobian $\frac{d\mathbf{\Phi}(\mathbf{\eta})}{d\mathbf{\eta}} = \left[\frac{\partial \varphi_i}{\partial \eta_k}\right]$ does not vanish in this vicinity.

Set of linearly independent vector functions $\{\mathbf{f_1}(\mathbf{\eta}) \ \cdots \ \mathbf{f_r}(\mathbf{\eta})\}$ is involutive if the Lie brackets of any (not necessarily distinct) functions $\mathbf{f_i}(\mathbf{\eta})$ and $\mathbf{f_j}(\mathbf{\eta})$ from this set are equal to a linear combination of functions from this set, i.e. there are functions such α_{ijk} that

$$[\mathbf{f_i}(\boldsymbol{\eta}),\mathbf{f_j}(\boldsymbol{\eta})]=\sum_{k=1}^{r}\alpha_{ijk}\mathbf{f_k}(\boldsymbol{\eta}). \tag{1.182}$$

The set of linearly independent constant vectors is always involutive. Indeed, the Lie brackets of two constant vectors are zero and trivially represented by combinations of the original vectors.

A set consisting of one vector is involutive, since the Lie brackets of two identical functions are equal to zero:

$$[\mathbf{f}(\boldsymbol{\eta}),\mathbf{f}(\boldsymbol{\eta})]=(\nabla\mathbf{f})\mathbf{f}-(\nabla\mathbf{f})\mathbf{f}=0. \tag{1.183}$$

A set r ($r<n$) of linearly independent n-dimensional vector functions $\{\mathbf{f_1}(\boldsymbol{\eta}) \quad \mathbf{f_2}(\boldsymbol{\eta}) \quad \cdots \quad \mathbf{f_r}(\boldsymbol{\eta})\}$ is integrable if there are nr independent scalar functions $\boldsymbol{\alpha_1}(\boldsymbol{\eta}) \quad \boldsymbol{\alpha_2}(\boldsymbol{\eta}) \quad \cdots \quad \boldsymbol{\alpha_{nr}}(\boldsymbol{\eta})$ satisfying the system of differential equations

$$\nabla\alpha_i(\boldsymbol{\eta})\mathbf{f_j}(\boldsymbol{\eta})=0, \quad i=1,2,\cdots,nr; \quad j=1,2,\cdots,r. \tag{1.184}$$

Scalar functions $\boldsymbol{\alpha_1}(\boldsymbol{\eta}) \quad \boldsymbol{\alpha_2}(\boldsymbol{\eta}) \quad \cdots \quad \boldsymbol{\alpha_{nr}}(\boldsymbol{\eta})$ are independent in some domain D if the vectors $\nabla\alpha_i(\boldsymbol{\eta}), \quad (i=1,2,\cdots,nr)$ are linearly independent in this domain. Note that if $r=n-1$, the independence of one single vector $\nabla\alpha_1(\boldsymbol{\eta})$ means that this vector is not equal to zero:

$$\left(\boldsymbol{\alpha_1}(\boldsymbol{\eta}) \quad \boldsymbol{\alpha_2}(\boldsymbol{\eta}) \quad \cdots \quad \boldsymbol{\alpha_{nr}}(\boldsymbol{\eta})\right)\neq 0.$$

A set r ($r<n$) of linearly independent n-dimensional vector functions $\{\mathbf{f_1}(\boldsymbol{\eta}) \quad \mathbf{f_2}(\boldsymbol{\eta}) \quad \cdots \quad \mathbf{f_r}(\boldsymbol{\eta})\}$ is integrable only if it is involutive. This statement is used to evaluate the integrability of nonlinear differential equations.

As an example, let us estimate the integrability of the system of differential equations

$$\begin{aligned} &2\eta_3\frac{\partial\alpha}{\partial\eta_1}-\frac{\partial\alpha}{\partial\eta_2}=0;\\ &-\eta_1\frac{\partial\alpha}{\partial\eta_1}-2\eta_2\frac{\partial\alpha}{\partial\eta_2}+\eta_3\frac{\partial\alpha}{\partial\eta_3}=0, \end{aligned} \tag{1.185}$$

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where $\alpha = \alpha(\eta_1, \eta_2, \eta_3)$ is an unknown function.

It is necessary to determine whether this system can be solved.

Let us represent the system (1.185) as follows

$$\frac{d\boldsymbol{\alpha}}{d\boldsymbol{\eta}}\mathbf{f_1} = 0;$$
$$\frac{d\boldsymbol{\alpha}}{d\boldsymbol{\eta}}\mathbf{f_2} = 0, \qquad (1.186)$$

where $\mathbf{f_1(\eta)} = \begin{pmatrix} f_{11} \\ f_{12} \\ f_{13} \end{pmatrix}\begin{pmatrix} 2\eta_3 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{f_2(\eta)} = \begin{pmatrix} f_{21} \\ f_{22} \\ f_{23} \end{pmatrix}\begin{pmatrix} -\eta_1 \\ -2\eta_2 \\ \eta_3 \end{pmatrix}$.

To answer the question whether a given system can be solved, according to the Frobenius theorem, it is sufficient to check the involutivity of the set $\mathbf{f_1}, \mathbf{f_2}$.

The Lie brackets for this set are

$$[\mathbf{f_1}, \mathbf{f_2}] = \frac{d\mathbf{f_2}}{d\boldsymbol{\eta}}\mathbf{f_1} - \frac{d\mathbf{f_1}}{d\boldsymbol{\eta}}\mathbf{f_2} =$$
$$= \begin{pmatrix} \frac{\partial f_{21}}{\partial \eta_1} & \frac{\partial f_{21}}{\partial \eta_2} & \frac{\partial f_{21}}{\partial \eta_3} \\ \frac{\partial f_{22}}{\partial \eta_1} & \frac{\partial f_{22}}{\partial \eta_2} & \frac{\partial f_{22}}{\partial \eta_3} \\ \frac{\partial f_{23}}{\partial \eta_1} & \frac{\partial f_{23}}{\partial \eta_2} & \frac{\partial f_{23}}{\partial \eta_3} \end{pmatrix}\begin{pmatrix} f_{11} \\ f_{12} \\ f_{13} \end{pmatrix} - \begin{pmatrix} \frac{\partial f_{11}}{\partial \eta_1} & \frac{\partial f_{11}}{\partial \eta_2} & \frac{\partial f_{11}}{\partial \eta_3} \\ \frac{\partial f_{12}}{\partial \eta_1} & \frac{\partial f_{12}}{\partial \eta_2} & \frac{\partial f_{12}}{\partial \eta_3} \\ \frac{\partial f_{13}}{\partial \eta_1} & \frac{\partial f_{13}}{\partial \eta_2} & \frac{\partial f_{13}}{\partial \eta_3} \end{pmatrix}\begin{pmatrix} f_{21} \\ f_{22} \\ f_{23} \end{pmatrix} =$$
$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2\eta_3 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\eta_1 \\ -2\eta_2 \\ \eta_3 \end{pmatrix} =$$
$$= \begin{pmatrix} -4\eta_3 \\ 2 \\ 0 \end{pmatrix}. \qquad (1.187)$$

Lie brackets of each pair of functions from a set $\mathbf{f_1}$, $\mathbf{f_2}$ can be represented as

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linear combinations of functions of this set as follows:

$$[\mathbf{f}_1,\mathbf{f}_2] = -2\mathbf{f}_1 + 0\mathbf{f}_2;$$
$$[\mathbf{f}_2,\mathbf{f}_1] = -[\mathbf{f}_1,\mathbf{f}_2] = 2\mathbf{f}_1 - 0\mathbf{f}_2;$$
$$[\mathbf{f}_1,\mathbf{f}_1] = [\mathbf{f}_2,\mathbf{f}_2] = 0\mathbf{f}_1 + 0\mathbf{f}_2. \quad (1.188)$$

Thus, the set $\{\mathbf{f}_1 \quad \mathbf{f}_2\}$ is involutive and, therefore, the considered system is integrable.

1.4.4. Linearization by state feedback

Consider the nonlinear system

$$p\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})\mathbf{u}, \quad (1.189)$$

where $\mathbf{f}(\boldsymbol{\eta}), \mathbf{g}(\boldsymbol{\eta})$ are smooth vector functions.

We will assume that the free movement of the system (1.189), described by the equation

$$p\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}), \quad (1.190)$$

is stable. The origin of the coordinate system is its equilibrium point $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.

Let us introduce a new control input, defined by the expression

$$\nu = w(u + \varphi(\boldsymbol{\eta})). \quad (1.191)$$

If there is an inverse transformation w^{-1} that allows to find the control input u using a known control ν

$$w^{-1}(\nu) = w^{-1}(w(u + \varphi(\boldsymbol{\eta})));$$
$$w^{-1}(\nu) = u + \varphi(\boldsymbol{\eta});$$
$$u = w^{-1}(\nu) - \varphi(\boldsymbol{\eta}), \quad (1.192)$$

then in the system (1.189) by selecting functions w and φ it is possible to compensate for the nonlinearity $\mathbf{f}(\boldsymbol{\eta}), \mathbf{g}(\boldsymbol{\eta})$ of the control object.

A system (1.189) is linearized by state feedback if there are diffeomorphism

$\hat{\boldsymbol{\eta}} = \mathbf{T}(\boldsymbol{\eta})$ and feedback transformations $u = \alpha(\boldsymbol{\eta}) + \beta(\boldsymbol{\eta})\nu$ such that the equation (1.189) takes the form

$$p\hat{\boldsymbol{\eta}} = \mathbf{A}\hat{\boldsymbol{\eta}} + \mathbf{M}\mathbf{v} \tag{1.193}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{1.194}$$

A linearized system of equations (1.193) whose coefficients are determined by matrices (1.194) is called a system of equations in Brunovsky controlled form. Any fully controllable linear time-invariant system can be transformed to this form.

For a nonlinear object (1.189), the controllability matrix takes the form

$$\mathbf{Y} = \begin{pmatrix} \mathbf{g} & ad_f\mathbf{g} & ad_f^2\mathbf{g} & \cdots & ad_f^{n-1}\mathbf{g} \end{pmatrix}. \tag{1.195}$$

Let us show that the matrix (1.195) is a generalization of the well-known controllability matrix for linear systems $\mathbf{Y} = \begin{pmatrix} \mathbf{M} & \mathbf{AB} & \mathbf{A}^2\mathbf{M} & \cdots & \mathbf{A}^n\mathbf{M} \end{pmatrix}$.

To do this, we introduce the following notations $\mathbf{f}(\boldsymbol{\eta}) = \mathbf{A}\boldsymbol{\eta},\ \mathbf{g}(\boldsymbol{\eta}) = \mathbf{M}$ and define the Lie brackets

$$ad_f\mathbf{g} = \frac{d\mathbf{g}}{d\boldsymbol{\eta}}\mathbf{f} - \frac{d\mathbf{f}}{d\boldsymbol{\eta}}\mathbf{g} = -\frac{d\mathbf{A}\boldsymbol{\eta}}{d\boldsymbol{\eta}}\mathbf{M} = -AM;$$

$$ad_f^2\mathbf{g} = -\frac{d\mathbf{f}}{d\boldsymbol{\eta}}ad_f\mathbf{g} = -\frac{d\mathbf{A}\boldsymbol{\eta}}{d\boldsymbol{\eta}}ad_f\mathbf{g} = (-A)(-AM) = A^2M;$$

$$ad_f^{n-1}\mathbf{g} = -\frac{d\mathbf{f}}{d\boldsymbol{\eta}}ad_f^{n-2}\mathbf{g} = (-1)^{n-2}\frac{d\mathbf{A}\boldsymbol{\eta}}{d\boldsymbol{\eta}}ad_f^{n-2}\mathbf{g} = (-1)^{n-1}A^{n-1}M. \tag{1.196}$$

Taking into account the fact that the " –" sign does not affect the rank of the matrix $\mathbf{Y}$, the obtained relations reproduce the columns of the well-known controllability matrix for a linear system.

Based on the previously found first-order Lie brackets for a nonlinear object (1.156), we construct a controllability matrix

$$\mathbf{Y} = \begin{pmatrix} \mathbf{g} & ad_f \mathbf{g} \end{pmatrix}. \tag{1.197}$$

Substituting the function $\mathbf{g}(\boldsymbol{\eta}) = \begin{pmatrix} 0 & m_2 \end{pmatrix}^T$ and the value of Lie brackets $L_f \mathbf{g} = -\begin{pmatrix} 2a_{12}m_2\eta_2 & a_{21}m_2\eta_1 + a_{22}m_2 \end{pmatrix}^T$ into the expressions (1.197), we obtain the controllability matrix

$$\mathbf{Y} = \begin{pmatrix} 0 & -2a_{12}m_2\eta_2 \\ m_2 & -a_{21}m_2\eta_1 - a_{22}m_2 \end{pmatrix}. \tag{1.198}$$

Summarizing the above material, we can state that a nonlinear system (1.189) can be linearized by state feedback in a vicinity Ω of the origin if and only if in this vicinity the controllability matrix $\mathbf{Y}$ has rank n, the determinant of the matrix $\mathbf{Y}$ is nonzero throughout the entire vicinity Ω (though it may be zero at the origin), and the set $\left\{ \mathbf{g} \quad ad_f \mathbf{g} \quad ad_f^2 \mathbf{g} \quad \cdots \quad ad_f^{n-2} \mathbf{g} \right\}$ composed of $n-1$ columns of the controllability matrix, is involutive.

Suppose there are transformations of state

$$\hat{\boldsymbol{\eta}} = \mathbf{T}(\boldsymbol{\eta}) \tag{1.199}$$

and control

$$u = \alpha(\boldsymbol{\eta}) + \beta(\boldsymbol{\eta})\nu \tag{1.200}$$

such that the variables $\hat{\boldsymbol{\eta}}$ and ν satisfy a system of linear differential equations in Brunovsky form

$$p\hat{\eta}_i = \hat{\eta}_{i+1}; \; p\hat{\eta}_n = \nu, \quad i \in [1, n-1]. \tag{1.201}$$

Taking into account the transformation (1.199) and equations (1.189), the equations of motion (1.201) will take the form

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$$p\hat{\eta}_i = \frac{\partial T_i}{\partial \boldsymbol{\eta}}\left(\mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u\right) = T_{i+1}, \quad i \in [1, n-1]$$
$$p\hat{\eta}_n = \frac{\partial T_i}{\partial \boldsymbol{\eta}}\left(\mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u\right)\nu. \tag{1.202}$$

For a system with single control input, which is included in the last equation of motion of the object, the functions $g_i(\boldsymbol{\eta}) = 0, i < n$ and $g_n(\boldsymbol{\eta}) \neq 0, i < n$, therefore the following expressions will be valid

$$\frac{\partial T_i}{\partial \boldsymbol{\eta}}\mathbf{g} = 0,\ i < n \quad \frac{\partial T_n}{\partial \boldsymbol{\eta}}\mathbf{g} \neq 0 \tag{1.203}$$

The relationship (1.203) allows us to write the Lie derivatives

$$L_g T_i = 0,\ \ i < n \quad L_g T_n \neq 0. \tag{1.204}$$

Taking into account the expressions (1.203), we can represent the first equation of the system (1.202) as follows

$$p\hat{\eta}_i = \frac{\partial T_i}{\partial \boldsymbol{\eta}}\left(\mathbf{f}(\boldsymbol{\eta})\right) = T_{i+1}, \quad i \in [1, n-1] \tag{1.205}$$

and get from it the relationship between the transformation functions

$$L_f T_i = T_{i+1}, \tag{1.206}$$

from which

$$\nabla T_1 ad_f^k \mathbf{g} = 0, k \in [0, n-2]$$
$$\nabla T_1 ad_f^{n-1} \mathbf{g} \neq 0. \tag{1.207}$$

The expressions (1.207) allow us to conclude that the system of vectors $\left\{\mathbf{g} \quad ad_f\mathbf{g} \quad ad_f^2\mathbf{g} \quad \cdots \quad ad_f^{n-2}\mathbf{g}\right\}$ is linearly independent and involutive.

Involutiveness follows from the existence of a function T_1 that satisfies the first $(n-1)$ equation of the system (1.207). Linear independence is explained by the difference between the first equations of a system (1.207) and the last equation of the same system.

Recall that the involutiveness of a set $\left\{\mathbf{g} \quad ad_f\mathbf{g} \quad ad_f^2\mathbf{g} \quad \cdots \quad ad_f^{n-2}\mathbf{g}\right\}$ means the existence of a transformation function $T_1(\boldsymbol{\eta})$, which is one of the components of the transformation vector $\mathbf{T}(\boldsymbol{\eta})$. Taking into account and substituting the previously found values of the Lie derivatives (1.206) into the expression (1.204), we write the following equality

$$L_g T_n = L_g L_f T_{n-1} = \cdots = L_g L_f^{n-1} T_1 = 0. \tag{1.208}$$

This equality allows us to recursively find all remaining components of the transformation vector $\mathbf{T}(\boldsymbol{\eta})$ by differentiating the known transformation function T_1

$$\begin{aligned}
pT_1 &= \frac{\partial T_1}{\partial \boldsymbol{\eta}}\frac{d\boldsymbol{\eta}}{dt} = \frac{\partial T_1}{\partial \boldsymbol{\eta}}(\mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u) = L_f T_1 + L_g T_1 u = T_2;\\
pT_2 &= \frac{\partial T_2}{\partial \boldsymbol{\eta}}\frac{d\boldsymbol{\eta}}{dt} = \frac{\partial T_2}{\partial \boldsymbol{\eta}}(\mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u) = L_f T_2 + L_g T_2 u =\\
&= T_3 + L_g L_f T_1 u = T_3;\\
&\vdots\\
pT_n &= \frac{\partial T_n}{\partial \boldsymbol{\eta}}\frac{d\boldsymbol{\eta}}{dt} = \frac{\partial T_n}{\partial \boldsymbol{\eta}}(\mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u) = L_f T_n + L_g T_n u =\\
&= L_f^n T_1 + L_g L_f^{n-1} T_1 u = T_n.
\end{aligned} \tag{1.209}$$

Taking into account the obtained derivatives, the system (1.189) can be represented as follows

$$\begin{aligned}
p\hat{\eta}_i &= \hat{\eta}_{i+1};\, i \in [1, n-1];\\
p\hat{\eta}_n &= L_f^n T_1 + L_g L_f^{n-1} T_1 u.
\end{aligned} \tag{1.210}$$

Last equation by substitution

$$u = \frac{1}{L_g L_g^{n-1} T_1}\left(-L_f^n T_1 + \nu\right), \tag{1.211}$$

Can be reduced to form

$$p\hat{\eta}_n = \nu. \tag{1.212}$$

Thus, the sufficiency of the theorem is proved and the transformation of an arbitrary nonlinear system into a system of linear equations in Brunovsky form is found.

Let us analyze the controllability matrix obtained in the previous example.

• The matrix $\mathbf{Y}$ has a rank corresponding to the order of the control object, i.e. $\operatorname{rank}(\mathbf{Y})=2$.

• The determinant of the matrix $\mathbf{Y}$ is found by the formula

$$\det(\mathbf{Y}) = 2a_{12}m_2^2\eta_2 \tag{1.213}$$

and is equal to zero only at $\eta_2 = 0$.

• Since the set $\left\{\mathbf{g} \quad ad_f\mathbf{g} \quad ad_f^2\mathbf{g} \quad \cdots \quad ad_f^{n-2}\mathbf{g}\right\}$ consists of $n-1$ columns of the controllability matrix, then in the considered case of a second-order object, it is simplified to a set containing only one element $\{\mathbf{g}\}$. Taking into account that the vector $\mathbf{g}$ contains only constant coefficients, this set is involutive.

Using the considerations reflected in the proof of sufficiency, we can formulate an algorithm for linearization by state feedback:

• For a given nonlinear system, an observability matrix $\mathbf{Y} = \left(\mathbf{g} \quad ad_f\mathbf{g} \quad ad_f^2\mathbf{g} \quad \cdots \quad ad_f^{n-1}\mathbf{g}\right)$ is constructed.

• Based on the observability matrix $\mathbf{Y}$, the possibility of using state feedback transformations is determined.

• If the observability matrix $\mathbf{Y}$ has a non-zero determinant and the set $\left\{\mathbf{g} \quad ad_f\mathbf{g} \quad ad_f^2\mathbf{g} \quad \cdots \quad ad_f^{n-2}\mathbf{g}\right\}$ is involutive, then from the relations

$$\begin{aligned} &\nabla T_1 ad_f^i\mathbf{g} = 0, \ \ i \in [0, n-2]; \\ &\nabla T_1 ad_f^{n-1}\mathbf{g} \neq 0 \end{aligned} \tag{1.214}$$

the transformation function T_1 is found.

• Other vector components $\mathbf{T}(\boldsymbol{\eta})$ are determined

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$$T_{i+1} = L_f T_i = L_f^i T_1 \tag{1.215}$$

and the control input transformation is found

$$u = \frac{1}{L_g L_f^{n-1} T_1}\left(-L_f^n T_1 + \nu\right) \tag{1.216}$$

• The original nonlinear system is reduced to a controlled Brunovsky form (1.201).

As an example, let us linearize the nonlinear system

$$\begin{aligned} p\eta_1 &= a_{12}\eta_2^2; \\ p\eta_2 &= a_{21}\eta_1\eta_2 + a_{22}\eta_2 + m_2 u, \end{aligned} \tag{1.217}$$

the controllability matrix for which was constructed in the previous example.

In this case, the relations (1.214) will take the form

$$\begin{aligned} &\nabla T_1 \mathbf{g} = \begin{pmatrix} \frac{\partial T_1}{\partial \eta_1} & \frac{\partial T_1}{\partial \eta_2} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \\ &= \begin{pmatrix} \frac{\partial T_1}{\partial \eta_1} & \frac{\partial T_1}{\partial \eta_2} \end{pmatrix} \begin{pmatrix} 0 \\ m_2 \end{pmatrix} = m_2 \frac{\partial T_1}{\partial \eta_2} = 0; \\ &\nabla T_1 ad_f \mathbf{g} = \begin{pmatrix} \frac{\partial T_1}{\partial \eta_1} & \frac{\partial T_1}{\partial \eta_2} \end{pmatrix} \begin{pmatrix} (ad_f \mathbf{g})_1 \\ (ad_f \mathbf{g})_2 \end{pmatrix} = \\ &= \begin{pmatrix} \frac{\partial T_1}{\partial \eta_1} & \frac{\partial T_1}{\partial \eta_2} \end{pmatrix} \left(-\begin{pmatrix} 2a_{12} m_2 \eta_2 \\ a_{21} m_2 \eta_1 + a_{22} m_2 \end{pmatrix} \right) = 2a_{12} m_2 \eta_2 \frac{\partial T_1}{\partial \eta_1} \neq 0. \end{aligned} \tag{1.218}$$

The first expression of the relations (1.218) indicates that the function T_1 does not depend on the coordinate η_2. The second relation allows us to define the function T_1 as follows

$$T_1 = f(\eta_1). \tag{1.219}$$

The simplest candidate function T_1 is the function

$$T_1 = \eta_1. \tag{1.220}$$

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Using the function T_1, all other components of the transformation vector $\mathbf{T}(\boldsymbol{\eta})$ can be found. For the considered second-order dynamic object, the function T_2 must be determined

$$T_2 = L_f T_1 = \nabla T_1 \mathbf{f} = \begin{pmatrix} \frac{\partial T_1}{\partial \eta_1} & \frac{\partial T_1}{\partial \eta_2} \end{pmatrix} \mathbf{f} =$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a_{12}\eta_2^2 \\ a_{21}\eta_1\eta_2 + a_{22}\eta_2 \end{pmatrix} = a_{12}\eta_2^2. \tag{1.221}$$

Taking into account the found functions T_1 and T_2, the transformation vector $\mathbf{T}$ can be written as follows

$$\mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ a_{12}\eta_2^2 \end{pmatrix}, \tag{1.222}$$

i.e., the new state variables, which are interconnected by differential equations in controlled Brunovsky form, are determined through the old state variables by dependencies

$$\hat{\eta}_1 = T_1 = \eta_1;$$
$$\hat{\eta}_2 = T_2 = a_{12}\eta_2^2. \tag{1.223}$$

Now let us define the transformation of the control input (1.216), which for the linearizable object will take the form

$$u = \frac{1}{L_g L_f T_1}\left(-L_f^2 T_1 + \nu\right). \tag{1.224}$$

Let us find the Lie derivatives $L_g L_f T_1$ and $L_f^2 T_1$ included in the linearizing control input (1.224). To do this, using the expression (1.221), we write the Lie derivative $L_g L_f T_1$

$$L_g L_f T_1 = L_g T_2 = \nabla T_2 \mathbf{g} = \begin{pmatrix} \frac{\partial T_2}{\partial \eta_1} & \frac{\partial T_2}{\partial \eta_2} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 2a_{12}\eta_2 \end{pmatrix} \begin{pmatrix} 0 \\ m_2 \end{pmatrix} = 2a_{12}m_2\eta_2 \qquad (1.225)$$

and Lie derivative $L_f^2 T_1$

$$L_f^2 T_1 = L_f T_2 = \nabla T_2 \mathbf{f} = \begin{pmatrix} \frac{\partial T_2}{\partial \eta_1} & \frac{\partial T_2}{\partial \eta_2} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 2a_{12}\eta_2 \end{pmatrix} \begin{pmatrix} a_{12}\eta_2^2 \\ a_{21}\eta_1\eta_2 + a_{22}\eta_2 \end{pmatrix} =$$

$$= 2a_{12}\eta_2 (a_{21}\eta_1\eta_2 + a_{22}\eta_2). \qquad (1.226)$$

i.e., the desired linearizing control input that brings the original nonlinear object to the form

$$p\hat{\eta}_1 = \hat{\eta}_2; \quad p\hat{\eta}_1 = \nu, \qquad (1.227)$$

is determined by the expression

$$u = \frac{1}{2a_{12}m_2\eta_2} \left(-2a_{12}\eta_2 (a_{21}\eta_1\eta_2 + a_{22}\eta_2) + \nu \right) =$$

$$= -\frac{-2a_{12}m_2\eta_2 (a_{21}\eta_1\eta_2 + a_{22}\eta_2)}{2a_{12}m_2\eta_2} + \frac{\nu}{2a_{12}m_2\eta_2} =$$

$$= -\frac{(a_{21}\eta_1\eta_2 + a_{22}\eta_2)}{m_2} + \frac{\nu}{2a_{12}m_2\eta_2} =$$

$$= \frac{1}{m_2} \left(-(a_{21}\eta_1\eta_2 + a_{22}\eta_2) + \frac{\nu}{2a_{12}\eta_2} \right). \qquad (1.228)$$

Analysis of the expression (1.228) indicates that the first component of the control input u compensates for the nonlinearity of the second equation of the original nonlinear object, and the second component contains the derivative of the right-hand side of the first equation of the control object and thus compensates for its nonlinearity.

1.4.5. Linearization by output feedback

Let us consider a nonlinear system in which the measured output is not the state vector $\boldsymbol{\eta}$, but some output variable y. The components of the state vector and the output variable are related to each other by the observability equation

$$p\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u,\ y = h(\boldsymbol{\eta}) \qquad (1.229)$$

Linearization by output feedback is a transformation of a nonlinear system (1.229), which allows the relationship between the original variable y and the control input u to be described by a linear dependency.

Let's consider the principle of linearization by output feedback for a dynamic object

$$\begin{aligned} p\eta_1 &= a_{11}\eta_1^2 + a_{12}\eta_1\eta_2; \\ p\eta_2 &= a_{21}\eta_1\eta_2 + a_{22}\eta_2 + m_2 u, \\ y &= a_{12}\eta_1\eta_2. \end{aligned} \qquad (1.230)$$

We will differentiate the variable y as many times as necessary to obtain a linear relationship between the j-th derivative of the original variable y and the control input u

$$\begin{aligned} py &= p(a_{12}\eta_1\eta_2) = a_{12}p\eta_1\eta_2 + a_{12}p\eta_2\eta_1 = \\ &= a_{12}(a_{11}\eta_1^2 + a_{12}\eta_1\eta_2)\eta_2 + a_{12}\eta_1(a_{21}\eta_1\eta_2 + a_{22}\eta_2 + m_2 u) = \\ &= (a_{11} + a_{21})a_{12}\eta_2\eta_1^2 + a_{12}^2\eta_2^2\eta_1 + a_{12}a_{22}\eta_1\eta_2 + a_{12}m_2\eta_1 u. \end{aligned} \qquad (1.231)$$

A linear relationship between the first derivative of the observed quantity and the control input has been achieved.

By denoting

$$\nu = py, \qquad (1.232)$$

we can find the control input u

$$u = \frac{\nu - (a_{11} + a_{21})a_{12}\eta_1^2\eta_2 - a_{12}^2\eta_1\eta_2^2 - a_{12}a_{22}\eta_1\eta_2}{m_2 a_{12}\eta_1} \qquad (1.233)$$

or

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$$u = \frac{\nu}{m_2 a_{12}\eta_1} - \frac{a_{11}+a_{21}}{m_2}\eta_1\eta_2 - \frac{a_{12}}{m_2}\eta_2^2 - \frac{a_{22}}{m_2}\eta_2. \quad (1.234)$$

Substituting the control input (1.234) into the equations (1.230)

$$\begin{aligned} &p\eta_1 = a_{11}\eta_1^2 + a_{12}\eta_1\eta_2; \\ &p\eta_2 = a_{21}\eta_1\eta_2 + a_{22}\eta_2 + \\ &+m_2\left(\frac{\nu}{m_2 a_{12}\eta_1} - \frac{a_{11}+a_{21}}{m_2}\eta_1\eta_2 - \frac{a_{12}}{m_2}\eta_2^2 - \frac{a_{22}}{m_2}\eta_2\right), \\ &y = a_{12}\eta_1\eta_2. \end{aligned} \quad (1.235)$$

Taking into account the control input (1.234), the second equation of the system (1.235) can be represented as follows

$$p\eta_2 = \frac{\nu}{a_{12}\eta_1} - a_{11}\eta_1\eta_2 - a_{12}\eta_2^2. \quad (1.236)$$

Thus, similarly to linearization by feedback along the state vector, linearization by output feedback completely compensates for the nonlinearity of the control object.

To summarize the above example, we can assert that to obtain a direct dependency between input and output, the observability equation should be differentiated.

The number of derivatives required to determine the desired relationship between output and input is called relative degree or relative order of the system. For a controllable system, the relative order r does not exceed the order n of the system.

Let us return to the consideration of the system (1.229) and find the control input that carries out linearization by feedback in a general form. To do this, we differentiate the output coordinate y with respect to time

$$\begin{aligned} &\frac{dy}{dt} = \frac{dh}{d\boldsymbol{\eta}}\frac{d\boldsymbol{\eta}}{dt} = \frac{dh}{d\boldsymbol{\eta}}(\mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u) = \nabla h(\mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u) = \\ &= \nabla h\mathbf{f}(\boldsymbol{\eta}) + \nabla h\mathbf{g}(\boldsymbol{\eta})u = L_f h + (L_g h)u. \end{aligned} \quad (1.237)$$

If in a vicinity of the origin of coordinates Ω the Lie derivative $L_g h$ becomes

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zero, then we perform repeated differentiation of the derivative $L_f h$, i.e.

$$\begin{aligned}\frac{d^2 y}{dt^2} &= \frac{d(L_f h)}{d\boldsymbol{\eta}}\frac{d\boldsymbol{\eta}}{dt} = \frac{d(L_f h)}{d\boldsymbol{\eta}}(\mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u) = \\ &= \nabla(L_f h)(\mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u) = \\ &= \nabla(L_f h)\mathbf{f}(\boldsymbol{\eta}) + \nabla(L_f h)\mathbf{g}(\boldsymbol{\eta})u = L_f^2 h + (L_g L_f h)u. \qquad (1.238)\end{aligned}$$

If $L_g L_f h = 0$, then we continue to differentiate the function $L_f^2 h$. The condition for the end of differentiation of the Lie derivative $L_f^r h$ is a non-zero value of the derivative $L_g L_f^{r-1} h$.

Using the found Lie derivatives $L_f^r h$ and $L_g L_f^{r-1} h$, we write the desired feedback transformation

$$u = \frac{1}{L_g L_f^{r-1} h}\left(-L_f^r h + \nu\right). \qquad (1.239)$$

By generalizing the concept of the relative order of a dynamical system, the following statement can be formulated:

A dynamical system (1.229) in the vicinity of the origin Ω has a relative degree r if the conditions are met in this region

$$\begin{aligned}&L_g L_f^i h = 0, i \in [0, r-2], \\ &L_g L_f^{r-1} h \neq 0. \end{aligned} \qquad (1.240)$$

This statement allows us to determine the relative order of any dynamic system.

For linear systems, it is identical to the difference between the degrees of the denominator and numerator of the transfer function. Let's show this with the following example.

Let the perturbed motion of the control object be described by linear differential equations

$$p\eta_1 = a_{12}\eta_2; \quad p\eta_2 = a_{21}\eta_1 + a_{22}\eta_2 + m_2 u, \tag{1.241}$$

and the observability equation has the following form

$$y = b_1\eta_1 + b_2\eta_2. \tag{1.242}$$

Let us write the equations (1.241) in vector form

$$p\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u, \tag{1.243}$$

where

$$\mathbf{f} = \begin{pmatrix} a_{12}\eta_2 \\ a_{21}\eta_1 + a_{22}\eta_2 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} 0 \\ m_2 \end{pmatrix}, \quad h = b_1\eta_1 + b_2\eta_2. \tag{1.244}$$

Differentiating the function h with respect to time, we find the first-order Lie derivatives

$$\frac{dh}{d\boldsymbol{\eta}}\frac{d\boldsymbol{\eta}}{dt} = \nabla h(\mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u) =$$

$$= \begin{pmatrix} b_1 & b_2 \end{pmatrix}\left[\begin{pmatrix} a_{12}\eta_2 \\ a_{21}\eta_1 + a_{22}\eta_2 \end{pmatrix} + \begin{pmatrix} 0 \\ m_2 \end{pmatrix}u\right],$$

$$L_g h = \begin{pmatrix} b_1 & b_2 \end{pmatrix}\begin{pmatrix} 0 \\ m_2 \end{pmatrix} = b_2 m_2 \neq 0;$$

$$L_f h = \begin{pmatrix} b_1 & b_2 \end{pmatrix}\begin{pmatrix} a_{12}\eta_2 \\ a_{21}\eta_1 + a_{22}\eta_2 \end{pmatrix} =$$

$$= b_1 a_{12}\eta_2 + b_2(a_{21}\eta_1 + a_{22}\eta_2). \tag{1.245}$$

Thus, in accordance with the conditions (1.240), the system (1.241) has a relative order equal to one. This conclusion corresponds to the result obtained by subtracting the orders of the denominator $\alpha = 2$ and numerator $\beta = 1$ of the transfer function

$$W(p) = \frac{b_1 + b_2 p}{p^2 + a_{22}p + a_{12}a_{21}}, \tag{1.246}$$

which is compiled on the basis of equations (1.241) and (1.242).

1.4.6. Linearization by feedback of non-affine systems

Classical feedback linearization was developed for affine control systems

$$p\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta})u, \qquad (1.247)$$

however, there are a number of dynamic objects and systems in which the trajectories of perturbed motion are determined by a nonlinear function of the control input

$$p\boldsymbol{\eta} = \mathbf{F}(\boldsymbol{\eta}, u). \qquad (1.248)$$

An example of such dynamic systems is electric drives, converters and power sources of which have nonlinear characteristics.

For such systems, the following approach has been developed [18], according to which a new function $\mathbf{F_1}(\boldsymbol{\eta}, u)$ is introduced, satisfying the relation

$$\mathbf{F_1}(\boldsymbol{\eta}, u) = \mathbf{F}(\boldsymbol{\eta}, u) - \mathbf{G_1}(\boldsymbol{\eta})u \qquad (1.249)$$

and the equations of motion (1.248) are written in the form

$$p\boldsymbol{\eta} = \mathbf{F_1}(\boldsymbol{\eta}, u) + \mathbf{G_1}(\boldsymbol{\eta})u. \qquad (1.250)$$

For a dynamic object, whose control input is related to the derivative of the internal state variable itself, it is convenient to take the function $\mathbf{G_1}(\boldsymbol{\eta})$ to be equal to one, i.e.

$$\mathbf{G_1}(\boldsymbol{\eta}) = \begin{pmatrix} 0 & 0 & \dots & 1 \end{pmatrix}^T. \qquad (1.251)$$

The further linearization procedure does not differ from that discussed earlier.

Let us consider some examples.

Let the dynamics of the control object be described by the nonlinear differential equation

$$p\eta = u^2. \qquad (1.252)$$

We can transform it as follows:

$$p\eta = u^2 - u + u \qquad (1.253)$$

and determine the linearizing control input

$$u = \nu - f(u), \tag{1.254}$$

where

$$f(u) = u^2 - u. \tag{1.255}$$

Substituting the function (1.255) into the expression (1.254), we obtain

$$u = \nu - u^2 + u. \tag{1.256}$$

From this:

$$0 = \nu - u^2 \tag{1.257}$$

or

$$\nu = u^2, \;\; u = \sqrt{\nu}\,. \tag{1.258}$$

Thus, an obvious replacement has been found, bringing the equation (1.252) to the controlled Brunovsky form

$$p\eta = v. \tag{1.259}$$

The given example demonstrates that for a first-order dynamic object that is non-affine in control, replacing the right-hand side of the equation of motion of the control object $f(\eta, u)$ in the form of a new control input v allows us to represent the dynamics of the original object in Brunovsky's controlled form. In this case, the control input applied to the object is determined by solving the nonlinear equation

$$v = f(\eta, u) \tag{1.260}$$

with respect to u.

Now let us consider a second-order nonlinear nonaffine object, whose dynamics is described by the equations:

$$\begin{aligned} p\eta_1 &= \eta_1\eta_2; \\ p\eta_2 &= -\eta_2 + \sqrt{U - \eta_1}. \end{aligned} \tag{1.261}$$

Let us introduce the functions

$$\mathbf{F}_1(\boldsymbol{\eta}, u) = \begin{pmatrix} \eta_1\eta_2 \\ -\eta_2 + \sqrt{U - \eta_1} - U \end{pmatrix}; \;\; \mathbf{G}_1(\boldsymbol{\eta}, u) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1.262}$$

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with which equations (1.261) can be represented in matrix form

$$p\boldsymbol{\eta} = \mathbf{F}_1(\boldsymbol{\eta}, u) + \mathbf{G}_1(\boldsymbol{\eta}, u)u. \qquad (1.263)$$

For the system (1.263) we find a transformation function $\mathbf{T} = (T_1 \quad T_2)$, whose components satisfy the relationships:

$$\nabla T_1 \mathbf{G_1} = 0;$$
$$\nabla T_1 ad_f \mathbf{G_1} \neq 0. \qquad (1.264)$$

Then

$$\nabla T_1 \mathbf{G_1} = \begin{pmatrix} \frac{\partial T_1}{\partial \eta_1} & \frac{\partial T_1}{\partial \eta_2} \end{pmatrix} \begin{pmatrix} G_{11} \\ G_{12} \end{pmatrix} =$$
$$= \begin{pmatrix} \frac{\partial T_1}{\partial \eta_1} & \frac{\partial T_1}{\partial \eta_2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\partial T_1}{\partial \eta_2} = 0 \qquad (1.265)$$

The expression (1.265) shows that the function T_1 does not depend on η_2. Thus, assuming that $T_1 = \eta_1$, we can determine

$$T_2 = \nabla T_1 \mathbf{F}_1(\boldsymbol{\eta}, u) = \begin{pmatrix} \frac{\partial T_1}{\partial \eta_1} & \frac{\partial T_1}{\partial \eta_2} \end{pmatrix} \begin{pmatrix} \eta_1 \eta_2 \\ -\eta_2 + \sqrt{U - \eta_1} - U \end{pmatrix} =$$
$$= (1 \quad 0) \begin{pmatrix} \eta_1 \eta_2 \\ -\eta_2 + \sqrt{U - \eta_1} - U \end{pmatrix} = \eta_1 \eta_2. \qquad (1.266)$$

Then the transformation function

$$\mathbf{T} = \begin{pmatrix} \eta_1 \\ \eta_1 \eta_2 \end{pmatrix}, \qquad (1.267)$$

and the control input that linearizes the object under consideration with feedback,

$$U = \frac{1}{L_g L_f T_1} \left(-L_f^2 T_1 + \nu \right), \qquad (1.268)$$

where

$$L_g L_f T_1 = L_g T_2 = \nabla T_2 \mathbf{G}_1(\boldsymbol{\eta}, u) = \begin{pmatrix} \eta_2 & \eta_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \eta_1; \tag{1.269}$$

$$L_f^2 T_1 = L_f T_2 = \nabla T_2 \mathbf{F}_1(\boldsymbol{\eta}, u) = \eta_1 \eta_2^2 - (U - \sqrt{U - \eta_1} + \eta_2)\eta_1. \tag{1.270}$$

Substituting the values of expressions (1.269) and (1.270) into the control algorithm (1.268), we obtain

$$U = \frac{1}{\eta_1}\left(-\eta_1 \eta_2^2 + (U - \sqrt{U - \eta_1} + \eta_2)\eta_1 + \nu\right) =$$
$$= -\eta_2^2 + U - \sqrt{U - \eta_1} + \eta_2 + \nu \tag{1.271}$$

or

$$0 = -\eta_2^2 - \sqrt{U - \eta_1} + \eta_2 + \nu. \tag{1.272}$$

The desired linearizing control input can be found from the equation (1.272)

$$\sqrt{U - \eta_1} = -\eta_2^2 + \eta_2 + \nu;$$
$$U - \eta_1 = (-\eta_2^2 + \eta_2 + \nu)^2;$$
$$U = (-\eta_2^2 + \eta_2 + \nu)^2 + \eta_1. \tag{1.273}$$

It should be noted that finding a linearizing control input by solving a nonlinear equation is a separate task that can only be solved analytically only in some cases. If it is impossible to determine U analytically, the problem of finding it must be solved numerically by the control system after finding the control input ν generated by the corresponding controller.

CHAPTER 2
METHODS OF OPTIMAL CONTROLS SYNTHESIS

2.1. Classical and modern methods of calculus of variations

2.1.1. Euler's method

The main task of the calculus of variations is to find functions that deliver an extremum to the functional. Mathematically, this task can be reduced to the analysis of solutions to the Euler equation, which are the required functions. In the calculus of variations, functions at which the functional reaches an extremum are called *extremals*.

Let us find a function that delivers the minimum to the functional

$$I = \int_{t_0}^{t_1} F(x, \dot{x}, t)dt \qquad (2.274)$$

with fixed boundary points of admissible functions $x(t_0) = x_0$ and $x(t_1) = x_1$. The function $F(x, \dot{x}, t)$ is assumed to be continuous and twice differentiable with respect to all arguments. The geometric interpretation of the problem is shown in Fig. 2.1.

It is necessary to find the equation of the curve passing through the boundary points $x(t_0)$ and $x(t_1)$, which, when substituted into the functional (2.1) would minimize it.

It is known that a necessary condition for an extremum is that the variation δI vanishes [19]. Let us apply this principle to the functional under consideration (2.1). Let us suppose that the extremum is reached at the function $x(t)$. Let us vary this function and determine the increment of the functional

$$\delta I = \int_{t_0}^{t_1} F(x + \delta x, \dot{x} + \delta \dot{x}, t)dt - \int_{t_0}^{t_1} F(x, \dot{x}, t)dt \, . \qquad (2.2)$$

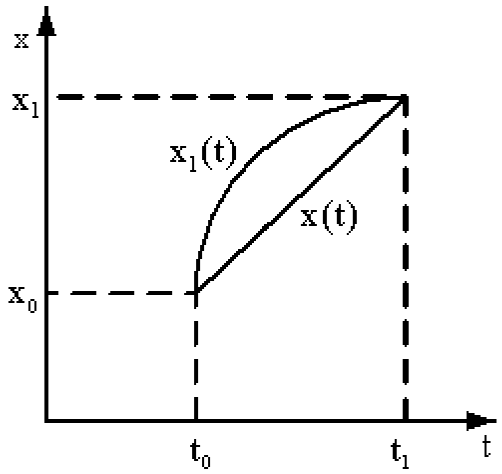

Fig. 2.1. Geometric interpretation of the variational problem

The variation of the argument is chosen in a way that $\delta x(t_0) = \delta x(t_1) = 0$, i.e. the varied line described by the function $x_1(t) = x(t) + \delta x(t)$ passed through the points $x(t_0)$ and $x(t_1)$.

Let us expand the varied function into a Taylor series

$$F(x+\delta x, \dot{x}+\delta\dot{x}, t) = F(x,\dot{x},t) + \left\{ \frac{\partial F(x,\dot{x},t)}{\partial x}\delta x + \frac{\partial F(x,\dot{x},t)}{\partial \dot{x}}\delta\dot{x} \right\} +$$
$$+\frac{1}{2}\left\{ \frac{\partial^2 F(x,\dot{x},t)}{\partial^2 x}\delta x^2 + 2\frac{\partial^2 F(x,\dot{x},t)}{\partial x \partial \dot{x}}\delta x \delta\dot{x} + \frac{\partial^2 F(x,\dot{x},t)}{\partial^2 \dot{x}}\delta\dot{x}^2 \right\} + R_n, \quad (2.3)$$

where R_n is the remainder of the third and higher orders of smallness.

The expansion term (2.3) in the first curly brackets is called the first variation, and it is linear. The expansion term in the second curly brackets is called the second variation, and it is nonlinear.

To determine the extremum of the functional, it is necessary to study the linear part of its increment, i.e. first variation . Due to the smallness of the components R_n, they can be neglected. Then the variation of the functional is determined by the expression

$$\delta I = \int_{t_0}^{t_1} \frac{\partial F(x,\dot{x},t)}{\partial x}\delta x dt + \int_{t_0}^{t_1} \frac{\partial F(x,\dot{x},t)}{\partial \dot{x}}\delta \dot{x} dt \quad . \tag{2.277}$$

Let us integrate by parts the second term in (2.4)

$$\int_{t_0}^{t_1} \frac{\partial F(x,\dot{x},t)}{\partial \dot{x}}\delta \dot{x} dt = \frac{\partial F(x,\dot{x},t)}{\partial \dot{x}}\delta x\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial F(x,\dot{x},t)}{\partial \dot{x}}\delta x dt. \tag{2.278}$$

The first term in the resulting expression is equal to zero, because $\delta[x(t_0)] = \delta[x(t_1)] = 0$ according to the conditions of the problem. Taking this into account, the expression for the variation of the functional takes the form:

$$\delta I = \int_{t_0}^{t_1} \left[\frac{\partial F(x,\dot{x},t)}{\partial x}\delta x - \frac{d}{dt}\frac{\partial F(x,\dot{x},t)}{\partial \dot{x}}\delta \dot{x} \right] \delta x dt \quad . \tag{2.279}$$

The integral (2.6) is equal to zero when the integrand is equal to zero, i.e.

$$\frac{\partial F(x,\dot{x},t)}{\partial x}\delta x - \frac{d}{dt}\frac{\partial F(x,\dot{x},t)}{\partial \dot{x}}\delta \dot{x} \equiv 0 \quad . \tag{2.280}$$

The expression (2.7) is Euler equation and extremals of functionals of the form (2.1) should be sought among the solutions of this equation.

When investigating the extremum of functionals of the form

$$I = \int_{t_0}^{t_1} F(x, \dot{x}, \cdots, x^{(n)}, t) dt, \tag{2.281}$$

depending on higher order derivatives, the Euler-Poisson equation should be used. This equation is similar to Euler equation (2.7), so we present it without derivation

$$F_x - \frac{d}{dt}F_{\dot{x}} + \frac{d^2}{dt^2}F_{\ddot{x}} + \cdots + (-1)^n \frac{d^n}{dt^n}F_{x^{(n)}} = 0, \tag{2.282}$$

where F_x, $F_{\dot{x}}$, …, $F_{x^{(n)}}$ are partial derivatives of the functional's integrand (2.8) with respect to the derivatives of the variable x from 0 to n-th order.

Equations (2.7) and (2.9) form the basis of classical variational calculus methods, which assume the continuity and linearity of the variations of functionals, the functions studied and their derivatives. This formulation of the variational problem has limited application in the theory of automatic control, as in most cases the control input belongs to a closed set, i.e. bounded in modulus.

Moreover, for real control objects, some phase coordinates must be constrained. Quite often, the extreme value of the chosen optimality criterion is achieved with discontinuous controls. Discontinuity points may also have derivatives of optimal trajectories. The position and number of discontinuity points are unknown in advance.

The noted circumstances have necessitated the development of modern methods of variational calculus that are free from these drawbacks.

2.1.2. Pontryagin's maximum principle

Among such methods is Pontryagin's maximum principle [20]. Let us consider its simplified proof from the perspective of its physical meaning.

Let us consider a control object whose perturbed motion is given by the equations

$$\frac{d\eta_i}{dt} = f_i(\eta_1 ..., \eta_n, U_1, ..., U_r), \;\; (i = 1, ..., n), \tag{2.283}$$

and the goal of control is to minimize functional

$$I = \int_0^\infty f_0(\eta_1 ..., \eta_n, U_1, ..., U_r)dt. \tag{2.284}$$

Let us introduce a new coordinate

$$\eta_0 = \int_0^\infty f_0(\eta_1 ..., \eta_n, U_1, ..., U_r)dt. \tag{2.285}$$

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Then

$$\frac{d\eta_0}{dt} = f_0(\eta_1..., \eta_n, U_1,..., U_r). \qquad (2.286)$$

Let us add the equation (2.13) to the system (2.10) and consider the movement of the control object in $n+1$-dimensional phase space

$$\frac{d\eta_j}{dt} = f_j(\eta_0, \eta_1..., \eta_n, U_1,..., U_r), \ \ (j = 0,1,..., n). \qquad (2.287)$$

Let us write the system (2.14) in vector form

$$\frac{d\tilde{\boldsymbol{\eta}}}{dt} = f(\tilde{\boldsymbol{\eta}}, \mathbf{U}), \qquad (2.288)$$

where $\tilde{\boldsymbol{\eta}}$ is the vector of state variables in (n + 1)-dimensional space in contrast to the n-dimensional vector η.

We will assume that the functions f are continuous and differentiable with respect to variables.

Let us formulate the control problem as follows. Among piecewise continuous functions satisfying the condition

$$U \leq 1, \qquad (2.289)$$

it is necessary to find the optimal control U^* that minimizes the functional (2.11) on the trajectories η^* of the system (2.14) from any initial position $\eta(0)$ to the origin $\eta(\infty) = 0$.

Let us assume that the functions η^* and U^* are known. Let us consider the change in optimal control over time (Fig. 2.2). Let us vary U^* on an infinitesimal interval ε, imposing a needle variation δU on it. The magnitude of the variation must be such that the varied control $U = U^* + \delta U$ satisfies the condition (2.16), i.e. it does not exceed the specified constraints. Since the duration ε of the needle variation is infinitesimal, even large values of δU have an infinitesimal effect on the subsequent movement of the control object.

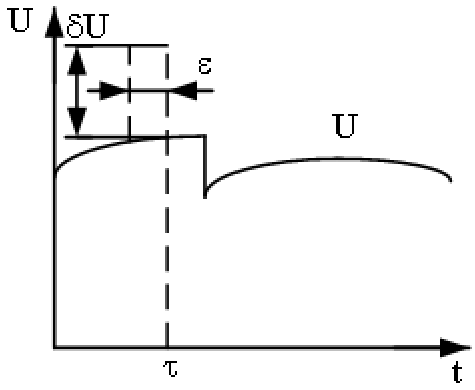

Fig. 2.2. Imposing a needle variation

The needle variation, which differs significantly from the smooth variation used in the classical variational calculus, allows one to expand the class of admissible functions and is the basis of the maximum principle. Since optimal control was applied to the object until the moment of time $t = \tau - \varepsilon$, it moved along the optimal trajectory $\tilde{\eta}^*$. As a result of varying the control over the interval $\tau - \varepsilon < t < \tau$, the subsequent movement $\tilde{\eta}$ at $t > \tau$ differs from the optimal one $\tilde{\eta}^*$ by the amount of trajectory variation $\delta\tilde{\eta} = \tilde{\eta} - \tilde{\eta}^*$. The value of $\delta\tilde{\eta}$ at a time $t = \tau$ can be determined as the product of the difference in the rates of change $\tilde{\eta}$ and $\tilde{\eta}^*$ the duration ε of the needle variation

$$\delta\tilde{\eta} = \varepsilon\left(\left(\frac{\tilde{\eta}}{dt}\right)_{t=\tau} - \left(\frac{\tilde{\eta}^*}{dt}\right)_{t=\tau}\right) = \varepsilon\Big(f(\tilde{\eta},U) - f(\tilde{\eta},U^*)\Big). \tag{2.290}$$

The variation $\delta\tilde{\eta}$ of the trajectory is infinitesimal, so the law of its change can be determined by solving the equations of motion in variations. Equations in variations are obtained from the basic equations (2.14) after replacing the variables η_j with

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$\eta_j + \delta\eta_j, (j = 0,1,\cdots,n)$ and Taylor series expansion in $\delta\eta_j$:

$$\frac{d(\eta_j + \delta\eta_j)}{dt} = f_j\left(\eta_0 + \delta\eta_0, \eta_1 + \delta\eta_1, ..., \eta_n + \delta\eta_n, u_1, ..., u_r\right) =$$
$$= f_j\left(\eta_0, \eta_1, ..., \eta_n, U_1, ..., U_r\right) +$$
$$\sum_{i=0}^{n} \delta\eta_i \frac{\partial f_j\left(\eta_0, \eta_1, ..., \eta_n, U_1, ..., U_r\right)}{\partial\eta_i} + R_n, \; j = 0,1,\cdots,n. \quad (2.291)$$

Discarding the term R_n of order of smallness greater than two and taking into account (2.14), we obtain equations in variations

$$\frac{d(\delta\eta_j)}{dt} = \sum \delta\eta_i \frac{\partial f_j(\tilde{\eta}, U)}{\partial\eta_i}, (j = 0,1,\cdots,n) \quad . \quad (2.292)$$

Among all the solutions to the equations (2.19), the value of the coordinate $\delta\eta_0$ at any moment in time t, $(\tau \le t \le \infty)$ is of greatest interest. This interest is quite understandable, since $\delta\eta_0$, according to (2.12), represents the variation of the functional δI that arose as a result of imposing a needle variation on the optimal control. Since only optimal control U^* can ensure the minimum value of the functional (2.11), any other control will lead to an increase of I. Hence,

$$\delta I = \delta\eta_0 \ge 0. \quad (2.293)$$

Let us introduce a vector $\tilde{\psi} = (\psi_0, \psi_1, \cdots, \psi_n)$ such that the scalar product of $\delta\tilde{\eta}$ and $\tilde{\psi}$ is equal to $-\delta\eta_0$, i.e.

$$\left\langle \delta\tilde{\eta}_0, \tilde{\psi}_0 \right\rangle_0 = -\delta\eta_0 \le 0. \quad (2.294)$$

As is known, the scalar product of two non-zero vectors is zero when these vectors are mutually perpendicular. Therefore, to fulfill the relationship (2.21), it is sufficient that the projections of the vectors $\delta\eta_i$ and $\psi_i \; (i = 1,\cdots,n)$ were mutually perpendicular, and the projections $\delta\eta_0$ and ψ_0 were counter-parallel.

The scalar product (2.21) for any non-optimal controls will be negative. Only with optimal control U^* it vanishes, reaching its maximum. This is the main idea of the maximum principle.

The needle variation of control ceases at $t = \tau$. At this point, the variation of the functional caused by the variation of control has reached its maximum value and remains unchanged for any time t, $(\tau \le t \le \infty)$.

Thus,

$$\langle \delta\tilde{\eta}(t), \tilde{\psi}(t) \rangle = \text{const}, \ (\tau \le t \le \infty) \tag{2.295}$$

from which it follows that

$$\frac{d}{dt}\langle \delta\tilde{\eta}(t), \tilde{\psi}(t) \rangle = 0, \ (\tau \le t \le \infty) \tag{2.296}$$

or

$$\left\langle \frac{d(\delta\tilde{\eta}(t))}{dt}, \tilde{\psi}(t) \right\rangle + \left\langle \delta\tilde{\eta}(t), \frac{d(\tilde{\psi}(t))}{dt} \right\rangle = 0, \quad (\tau \le t \le \infty). \tag{2.297}$$

Let us present equality (2.24) in expanded form

$$\sum_{j=0}^{n} \psi_j(t) \frac{d(\delta\eta_j(t))}{dt} + \sum_{i=0}^{n} \delta\eta_j(t) \frac{d(\psi_i(t))}{dt} = 0 \tag{2.298}$$

and substitute into it the values of the derivatives from the expression (2.19)

$$\sum_{j=0}^{n} \psi_j(t) \sum_{i=0}^{n} \delta\eta_i(t) \frac{\partial f_j(\tilde{\boldsymbol{\eta}}, U)}{\partial \eta_i} + \sum_{i=0}^{n} \delta\eta_i(t) \frac{d(\psi_i(t))}{dt} = 0. \tag{2.299}$$

Changing the order of summation over i and j in the first term of the expression (2.26), we obtain

$$\sum_{i=0}^{n} \delta\eta_i(t) \left(\sum_{j=0}^{n} \psi_j(t) \frac{\partial f_j(\tilde{\boldsymbol{\eta}}, U)}{\partial \eta_i} + \frac{d(\psi_i(t))}{dt} \right) = 0, \tag{2.300}$$

from which it follows that

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$$\frac{d(\psi_i(t))}{dt} = -\sum_{j=0}^{n} \psi_j(t) \frac{\partial f_j(\tilde{\boldsymbol{\eta}}, U)}{\partial \eta_i}, (i = 0, 1, \cdots, n) \quad . \quad (2.301)$$

Linear differential equations (2.28) are conjugate to the system (2.14).

Let us consider the inequality (2.21) after substituting the value $\delta\tilde{\eta}$ from (2.17) into it and dividing by ε:

$$\left\langle f(\tilde{\eta}, U), \tilde{\psi} \right\rangle - \left\langle f(\tilde{\eta}, U^*), \tilde{\psi} \right\rangle \le 0, \quad (2.302)$$

as well as the Hamilton function

$$H = \left\langle f(\tilde{\eta}, U), \tilde{\psi} \right\rangle. \quad (2.303)$$

From the expression (2.29) it follows that optimal control should deliver the maximum value of the function H. This is the essence of the maximum principle, the main provisions of which are quite clearly presented graphically for a second-order system (Fig. 2.3).

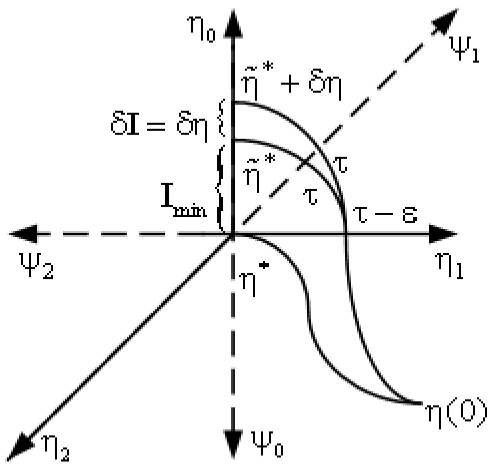

Fig. 2.3. Variation of the optimal trajectory

In the plane of two coordinates (η_1, η_2), the trajectory of the system under the influence of optimal control begins at point of initial disturbances $\eta(0)$ and ends at the origin. In the three-coordinate space (η_1, η_2, η_0), the optimal trajectory $\tilde{\eta}$ cuts off a segment on the axis η_0 equal to the minimum value $I_{\min}$ of the accepted quality criterion. Over the time interval $\tau - \varepsilon \le t \le \tau$, there is a variation $\delta\tilde{\eta}$ of the trajectory $\tilde{\eta}$ caused by imposing a needle variation on the optimal control (Fig. 2.3). Further movement of the system takes place along the trajectory $\tilde{\eta} + \delta\eta$ that cuts off a segment $I_{\min} + \delta I$ on the axis η_0 after the end of the transition process. From the arrangement of the conjugate system's coordinate axes (ψ_0, ψ_1, ψ_2), it follows that the scalar product of the vectors $\delta\tilde{\eta}$ and $\tilde{\psi}$ at $|\psi_0| = 1$ is determined by the relationship

$$\langle \delta\tilde{\eta}, \tilde{\psi} \rangle = \langle \delta\tilde{\eta}_0, \tilde{\psi}_0 \rangle + \langle \delta\tilde{\eta}_1, \tilde{\psi}_1 \rangle + \langle \delta\tilde{\eta}_2, \tilde{\psi}_2 \rangle = -\delta\tilde{\eta}_0 \le 0. \quad (2.304)$$

Consequently, optimal control turns the inequality (2.31) into identity, i.e. delivers the maximum value to the product $\langle \delta\tilde{\eta}, \tilde{\psi} \rangle$, and can be determined from the maximum condition of the Hamiltonian function

$$\max_U H(\tilde{\eta}, \tilde{\psi}, U) = 0. \quad (2.305)$$

In many cases, it is impossible to find the explicit form of the optimal control from condition (2.32). Then the equations (2.14), the conjugate system (2.28) and the maximum conditions (2.32) form the boundary value problem of the maximum principle. This problem has a number of specific features that make it difficult to use standard numerical methods for solving boundary value problems. These features include discontinuities of functions $U_1, \cdots, U_r$ satisfying the maximum condition (2.32), their non-uniqueness, and the nonlinear nature of the dependence $U = U(\eta, \psi)$ when the condition is met (2.16)even in linear systems. In addition, a feature of solving optimization problems associated with the maximum principle, even in cases where it is possible to find an explicit form of optimal controls, is their poor convergence caused by the instability of the joint solution of systems (2.10) and (2.28) [15].

2.1.3. Dynamic programming

A very general method for solving optimal control problems, known as dynamic programming, was proposed by R. Bellman [21]. Let us consider the main provisions of this method.

Let us start by solving the problem of the optimal speed of transition of the control object (2.10) from the phase state $\eta(0)$ to the origin $\eta = 0$. Let us assume that the trajectory of such a transition exists for any initial disturbances and that the movement along it occurs in a minimum time under the influence of admissible controls U.

Let T denote the time during which the movement along the optimal trajectory is carried out. For simplicity, consider the movement of a second-order control object from an arbitrary starting point $\eta(0)$ to the origin $\eta = 0$. Let us assume that for some time $t_1 - t_0$ the control object moved from point $\eta(0)$ to point η_1 under the influence of arbitrary constant control $U = U_0$ along a non-optimal trajectory 1 (Fig. 2.4).

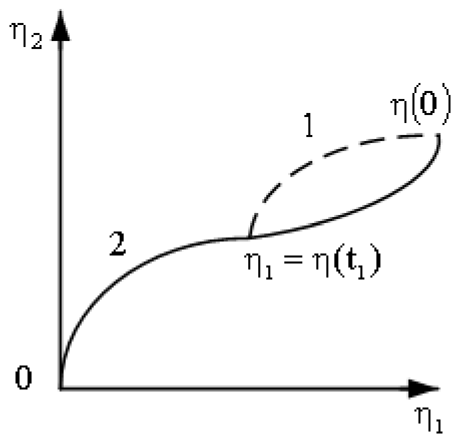

Fig. 2.4. Phase trajectory

Starting from point η_1, the object is transferred to optimal trajectory 2. The time of movement along the optimal trajectory T depends on the position of the starting point η_1 of transition from trajectory 1 to trajectory 2, i.e. is a function of the phase coordinates of the system

$$T = S(\eta(t)). \tag{2.306}$$

The function (2.33) is continuous and has continuous partial derivatives with respect to the coordinates of the phase space everywhere, except for the origin. Moving along the optimal trajectory, the object will take time equal to $T_1 = S(\eta(t_1))$ to move from the point η_1 to the origin. As a result, the transition from the point $\eta(0)$ to the origin along trajectories 1 and 2 will be completed in time $(t_1 - t_0) + S(\eta(t_1))$. If the movement from the point $\eta(0)$ had immediately occurred along the optimal trajectory 2, then the control object would have been transferred to the origin in the minimal time $T = S(\eta(t_0))$.

Hence,

$$S(\eta(t_0)) \le (t_1 - t_0) + S(\eta(t_1)). \tag{2.307}$$

Dividing both sides of the inequality (2.34) by the positive duration of the interval $t_1 - t_0$, we get

$$-\frac{S(\eta(t_1)) - S(\eta(t_0))}{t_1 - t_0} \le 1. \tag{2.308}$$

Let us move to the limit at $t_1 \to t_0$:

$$\lim_{t_1 \to t_0} \left[-\frac{d}{dt} S(\eta(t)) \right] \le 1. \tag{2.309}$$

The derivative (2.36) is calculated using the formula for the total derivative for the system (2.10)

$$-\frac{d}{dt} S(\eta(t)) = -\sum_{i=1}^{n} \frac{\partial S(\eta(t))}{\partial \eta_i} \dot{\eta}_i = -\sum_{i=1}^{n} \frac{\partial S(\eta(t))}{\partial \eta_i} f_i(\eta, U). \tag{2.310}$$

Then

$$-\sum_{i=1}^{n}\frac{\partial S(\eta(t))}{\partial \eta_i} f_i(\eta,U)\leq 1. \tag{2.311}$$

Obviously, inequality (2.38) turns into equality only with optimal control in terms of speed. In other words, optimal control delivers the maximum derivative (2.37) , which physically represents the rate of decrease of the transition time, i.e.

$$\max_U\left[-\sum_{i=1}^{n}\frac{\partial S(\eta)}{\partial \eta_i} f_i(\eta,U)\right]=1 \tag{2.312}$$

or

$$\min_U\left[\sum_{i=1}^{n}\frac{\partial S(\eta)}{\partial \eta_i} f_i(\eta,U)+1\right]=0. \tag{2.313}$$

Let us now consider the general control problem that is optimal in terms of minimizing the integral functional

$$I=\int_{t_0}^{t_1} f_0(\eta,U)dt. \tag{2.314}$$

The problem of optimal control of an object (2.10) is formulated as follows: from all permissible controls that transfer the representing point from its position $\eta(0)$ to the origin, it is required to choose such a control that minimizes the functional (2.41). Please note that if $f_0(\eta,U)=1$, the functional (2.41) takes the form

$$I=\int_0^T dt \;.$$

Therefore, the problem of optimal performance is a particular case of the general problem considered.

Let us assume that $f_0(\eta,U)>0$. This condition is satisfied for all integral quadratic functionals. Let us introduce a new time τ on each trajectory, related to the transition process time by the differential relationship

$$d\tau = f_0(\eta, U)dt\,. \tag{2.315}$$

In the new time, the functional (2.41) is transformed into

$$I = \int_{\tau_0}^{\tau_1} d\tau = \tau_1 - \tau_0, \tag{2.316}$$

and the problem posed is reduced to the previously considered problem of optimal performance.

Let U be a control that transfers the representing point from position $\eta(0)$ to position η_1, and $\eta(t)$ be the corresponding trajectory. Let us suppose that

$$\tau(t) = \int_{t_0}^{t_1} f_0(\eta, U)dt. \tag{2.317}$$

Differentiating the functional (2.44) with respect to time, we get

$$\frac{d\tau}{dt} = f_0(\eta, U). \tag{2.318}$$

The function $\tau(t)$ is continuous and monotonically increasing, since $f_0 > 0$. Therefore, there is an inverse function $t(\tau)$, for which the following value of the derivative is true:

$$\frac{dt}{d\tau} = \frac{1}{f_0(\eta(\tau), U(\tau))}\,. \tag{2.319}$$

Consequently, in the new time domain, the control object (2.10) is described by a system of differential equations

$$\frac{d\eta_i(t)}{dt} = \frac{d\eta_i(\tau)}{dt}\frac{dt(\tau)}{d\tau}\frac{f_i(\eta(\tau), U(\tau))}{f_0(\eta(\tau), U(\tau))}. \tag{2.320}$$

Substituting (2.47) in (2.37) and further into (2.40), we get

$$\min_U \left[\sum_{i=1}^{n} \frac{\partial S(\eta)}{\partial \eta_i} \frac{f_i(\eta, U)}{f_0(\eta, U)} + 1 \right] = 0 \tag{2.321}$$

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or

$$\min_U \left[\sum_{i=1}^{n} \frac{\partial S(\eta)}{\partial \eta_i} f_i(\eta, U) + f_0(\eta, U) \right] = 0. \qquad (2.322)$$

The expression (2.49) is called the Bellman functional equation, and $S(\eta)$ is the Bellman function.

To determine the optimal control U^* that minimizes expression (2.49), it is necessary to find the derivative

$$\sum_{i=1}^{n} \frac{\partial S(\eta)}{\partial \eta_i} \frac{\partial f_i(\eta, U)}{\partial U} + \frac{\partial f_0(\eta, U)}{\partial U} = 0 \qquad (2.323)$$

and solve (2.49) and (2.50) together.

2.1.4. The Lyapunov's second method

One of the most effective methods for studying motion stability is the direct Lyapunov method, which is often referred to as Lyapunov's second method. To reveal the essence of this method, let us consider some real functions $V(\eta) = V(\eta_1, \cdots, \eta_n)$ defined in the domain

$$\sum_{i=1}^{n} \eta_i^2 \le \mu, \qquad (2.324)$$

where μ is a positive constant.

It is assumed that in the domain (2.51) these functions are single-valued, continuous and vanish when all $\eta_1, \cdots, \eta_n$ are equal to zero, that is

$$V(0) = 0. \qquad (2.325)$$

If in the vicinity of the origin a function V can take only values of one sign besides zero, it is called constant sign (positive or negative, respectively). If a function of constant sign vanishes only at the origin of coordinates, then the function V is called sign-definite (correspondingly positive-definite or negative-definite). Such functions V are used to study motion stability and are called Lyapunov functions.

A sign-definite function has an extremum (a minimum for a positive-definite function and a maximum for a negative-definite function) at $\eta_1 = \cdots = \eta_n = 0$. A sign-definite function does not have an extremum at the origin.

Let us assume that the positive-definite function $V = V(\eta)$ is continuous along with its first order derivatives. Then at $\eta_1 = \cdots = \eta_n = 0$ it will have an isolated extremum, and all first-order partial derivatives at this point will be equal to zero

$$\left(\frac{\partial V}{\partial \eta_i}\right)_0 = 0, (i = 1, \cdots, n). \tag{2.326}$$

Let us expand the function $V(\eta)$ into the Maclaurin series

$$V = V(0) + \sum_{i=1}^{n}\left(\frac{\partial V}{\partial \eta_i}\right)_0 \eta_i + \frac{1}{2}\sum_{i,k=1}^{n}\left(\frac{\partial^2 V}{\partial \eta_i \partial \eta_k}\right)_0 \eta_i \eta_k + R_n, \tag{2.327}$$

where R_n are the higher order terms of expansion.

Taking into account the relations (2.52) and (2.53), we obtain

$$V = \frac{1}{2}\sum_{i,k=1}^{n} v_{ik}\eta_i\eta_k + R_n. \tag{2.328}$$

Here the constants $v_{ik} = v_{ki}$ are defined by the expression

$$v_{ik} = \left(\frac{\partial^2 V}{\partial \eta_i \partial \eta_k}\right)_0. \tag{2.329}$$

From the expression (2.55) it is evident that the expansion of a positive-definite function V into a Maclaurin series in powers $\eta_1, \cdots, \eta_n$ does not contain first-degree terms.

Thus, regardless of the higher-order terms, for sufficiently small modular values of η_i, the function V will be positive-definite if the quadratic form is positive-definite

$$\widetilde{V} = \frac{1}{2} \sum_{i,k=1}^{n} v_{ik} \eta_i \eta_k . \tag{2.330}$$

That is why in control theory the search for Lyapunov functions is carried out in the class of quadratic functions of the form (2.57).

To determine the positive-definiteness of quadratic forms (2.57), the Sylvester criterion is used [22]; according to which a quadratic form is positive-definite if all the main diagonal minors of the matrix of its coefficients v_{ik} are positive.

The direct Lyapunov's method [23] is based on the following two theorems.

• Lyapunov's theorem on stability of motion. If for the differential equations of perturbed motion (2.10) it is possible to find a positive-definite function V whose total time derivative would be a negative function of constant sign, then the unperturbed motion is stable.

• Lyapunov's theorem on asymptotic stability. If for the differential equations of perturbed motion (2.10) it is possible to find a positive-definite function V whose total time derivative would be a negative sign-determined function, then the unperturbed motion is asymptotically stable.

Physically, the Lyapunov's function can be identified with the excess energy stored by the system along the trajectories of the perturbed motion, compared to the energy stored along the trajectories of unperturbed motion. If the excess energy of the system constantly decreases, as evidenced by the negativeness of the derivative of the Lyapunov function, then the forces causing the deviation of the actual motion from the unperturbed one decrease. In this case, the system under study returns to the trajectories of unperturbed motion, regardless of the initial deviations.

Now let us demonstrate the conditions under which the main functional Bellman equation corresponds to the direct Lyapunov method.

Let us consider the equations (2.10) and functional (2.11). We will assume that there is a positive-definite Lyapunov function $V(\eta)$, and the integrand of the

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functional (2.11) $f_0(\eta,U)$ is a positive sign-definite or positive-definite function. Then, in accordance with the above Lyapunov theorems, a closed-loop control system, optimal in the sense of minimizing the functional (2.11), will be stable if the conditions are met

$$\frac{dV(\eta)}{dt}=\sum_{i=1}^{n}\frac{\partial V(\eta)}{\partial \eta_i}f_i(\eta,U)=-f_0(\eta,U) \tag{2.331}$$

or

$$\frac{dV(\eta)}{dt}=\sum_{i=1}^{n}\frac{\partial V(\eta)}{\partial \eta_i}f_i(\eta,U)+f_0(\eta,U)=0. \tag{2.332}$$

Thus, if a positive-definite or non-negative function is chosen as the integrand of the optimality criterion, in the process of solving variational problems the Bellman function can definitely be replaced by the Lyapunov function. This follows from the comparison of expressions (2.49) and (2.59).

2.1.5. Analytical design of regulators

The analytical solution to the problem of optimal stabilization of linear stationary objects with a quadratic quality functional [25] was proposed by A. M. Letov, who identified ways to overcome the difficulties of solving boundary value problems. This approach is called *analytical design of controllers* (ADC). Thanks to the clear formulation of the problem and constructive results, this method has become a common tool for synthesizing optimal controls for various classes of objects. Simultaneously with A. M. Letov, research in this areas was carried out by R. Kalman, and in foreign sources it became known as linear quadratic optimization [26]. Therefore, the ADC problem is also referred to as the Letov-Kalman problem. Each of these approaches has its own peculiarities, but both lead to similar results, which indicates the correctness of the synthesis problem formulation.

Let the perturbed motion of a generalized object of the n-th order be described by a system of linear or linearized differential equations of perturbed motion in the

Cauchy form

$$p\eta_i = \sum_{k=1}^{n} b_{ik}\eta_k + m_i U,\ i = 1,2,\cdots,n, \tag{2.333}$$

where b_{ik} are constant coefficients.

Among the set of piecewise smooth functions that satisfy the constraint (2.16), it is necessary to find the optimal control $U(\eta)$ that transfers the system (2.60) from the initial position $(\eta_1(0),\cdots,\eta_n(0))$ to the origin $(\eta_1(\infty)=0,\cdots,\eta_n(\infty)=0)$ in such a way that minimizes the integral functional

$$I = \int_0^\infty F(\eta,U)dt = \int_0^\infty \left(\sum_{i=1}^{n} w_i\eta_i^2 + cU^2 \right) dt \ . \tag{2.334}$$

Initially, this problem was reduced to the Lagrange conditional extremum problem. For this purpose, the system (2.60) is represented in the form

$$\varphi_i(\eta,U,t) = p\eta_i - \sum_{k=1}^{n} b_{ik}\eta_k - m_i U,\ i = 1,2,\cdots,n, \tag{2.335}$$

and the functional is introduced into consideration

$$I^* = \int_0^\infty \left(F(\eta,U) + \sum_{i=1}^{n} \lambda_i(t)\varphi_i(\eta,U,t) \right) dt = \int_0^\infty L(\eta,U,\lambda,t)dt, \tag{2.336}$$

where $\lambda_i(t)$ are the undetermined Lagrange multipliers.

Then, by solving Euler's equations, the extremals $\eta_i(t)$ and $\lambda_i(t)$ of the functional (2.63) are determined. Euler's equations for the function $L(\eta,U,\lambda,t)$ have the form

$$\frac{\partial L}{\partial \eta_i} - p\frac{\partial L}{\partial (p\eta_i)} = 0,\ (i = 1,\cdots,n); \tag{2.337}$$

$$\frac{\partial L}{\partial U} - p\frac{\partial L}{\partial (pU)} = 0. \tag{2.338}$$

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Optimal control as a function of the uncertain Lagrange multipliers is determined from the equation (2.65)

$$U(t)=\frac{1}{2c}\sum_{k=1}^{n} m_k\lambda_k(t). \tag{2.339}$$

Substituting (2.66) into (2.62), a system of $2n$ equations with $2n$ unknowns $\eta_i(t)$, $\lambda_i(t)$ is considered

$$\begin{aligned} p\eta_i &= \sum_{i,k}^{n} b_{ik}\eta_k + \frac{1}{2c}\sum_{k=1}^{n} m_i m_k \lambda_k(t); \\ p\lambda_i &= 2w_i\eta_i^2 - \sum_{k=1}^{n} \lambda_k b_{ik},\ (i=1,...,n). \end{aligned} \tag{2.340}$$

To solve this system of differential equations, it is necessary to determine the $2n$ roots of the characteristic equation of the system (2.67)

$$D(\mu)=\begin{vmatrix} b_{11}-\mu & \cdots & b_{1n} & \frac{m_1^2}{2c} & \cdots & \frac{m_1 m_n}{2c} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{nn}-\mu & \frac{m_n m_1}{2c} & \cdots & \frac{m_n^2}{2c} \\ -2w_1 & \cdots & 0 & -b_{11}-\mu & \cdots & -b_{n1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -2w_n & -b_{1n} & \cdots & -b_{nn}-\mu \end{vmatrix}, \tag{2.341}$$

which has the property that if μ is any root of it, then $-\mu$ is also a root [27]. To ensure the stability of a closed-loop system (2.60) with control (2.66), n roots with a negative real part are taken into account, and the remaining n roots lying in the right half-plane are neglected.

After this, the general solution of the system (2.67) is determined as a sum of exponential functions

$$\eta_i = \sum_{k=1}^{n} \Delta_i(\mu_k) C_k e^{\mu_k t}, \ (i = 1,...,n),$$
$$\lambda_i = \sum_{k=1}^{n} \Delta_{n+i}(\mu_k) C_k e^{\mu_k t}, \tag{2.342}$$

where Δ_i, Δ_{n+i} are the minors of the i-th or $(n+i)$-th element of the first row of the determinant (2.68), C_k are arbitrary constants.

To determine the optimal control algorithm as a function of the phase coordinates of the system (2.60), the uncertain Lagrange multipliers $\lambda_i(t)$ must be expressed through the variables $\eta_i(t)$. This is done by eliminating the exponential functions $C_k e^{\mu_k t}$ from the system (2.69). As a result, optimal control takes the form

$$U = \sum_{i=1}^{n} \beta_i \eta_i. \tag{2.343}$$

Control (2.70) was found without considering the restrictions (2.16). In closed area that satisfies the condition (2.16), the desired optimal control is determined by the expression

$$U = \begin{cases} 1 & \text{if} \quad \sum_{i=1}^{n} \beta_i \eta_i > 1; \\ \sum_{i=1}^{n} \beta_i \eta_i & \text{if} \quad \left| \sum_{i=1}^{n} \beta_i \eta_i \right| \le 1; \\ -1 & \text{if} \quad \sum_{i=1}^{n} \beta_i \eta_i < -1. \end{cases} \tag{2.344}$$

The expression (2.71) has a more compact form

$$U = \text{sat}\left(\sum_{i=1}^{n} \beta_i \eta_i \right), \tag{2.345}$$

where $\text{sat}(.)$ is a function equal to the argument when its absolute value is less than one, and becomes a sign function otherwise.

Let us represent the system (2.60) as follows

$$f_i = p\eta_i = \sum_{k=1}^{n} b_{ik}\eta_k + m_i U,\ i = 1,2,\cdots,n \tag{2.346}$$

and introduce an additional coordinate η_0, which satisfies the relation

$$f_0 = p\eta_0 = \sum_{i=1}^{n} w_i\eta_i^2 + cU^2. \tag{2.347}$$

The system, which is conjugated to the equations (2.73) and (2.74) can be represented as

$$p\varphi_i = -\sum_{j=0}^{n} \psi_j \frac{\partial f_j \cdot}{\partial \eta_i},\ (i = 0,\cdots,n) \tag{2.348}$$

The Hamilton function for systems (2.73) and (2.75) has the form

$$H = \sum_{i=0}^{n} f_i \psi_i. \tag{2.349}$$

The optimal control that minimizes the functional (2.63), delivers the maximum of the function (2.76) and is determined from the condition

$$\frac{\partial H}{\partial U} = 0, \tag{2.350}$$

from where, considering the fact that $\psi_0 = -1$, follows

$$U = \sum_{k=1}^{n} \frac{m_k}{2c}\psi_k. \tag{2.351}$$

Thus, optimal control depends on still unknown functions ψ_k and can be expressed through the phase coordinates of the control object if the dependence $\psi_k(\eta_i)$ is determined. To do this, it is necessary to join the conjugate system (2.75) to the system taking into account (2.78)

$$p\eta_i = \sum_{k=1}^{n} b_{ik}\eta_k + \sum_{k=1}^{n} \frac{m_i m_k}{2c}\psi_k ; i = 1,\cdots,n,$$
$$p\psi_i = -\sum_{j=0}^{n} \varphi_j \frac{\partial f_j}{\partial \eta_i}, i = 0,...,n. \tag{2.352}$$

System of equations (2.79) is identical to the system (2.67), so the further solution process does not differ from the one discussed above. As a result, an optimal control algorithm (2.72) is determined.

Optimal control $U(\eta_i)$, which belongs to the class of piecewise continuous functions, subject to the constraint (2.16), and delivering a minimum of the functional

$$I = \int_0^{\infty} F(\eta, U)dt = \int_0^{\infty} \left(\eta^T W \eta + cU^2\right)dt \tag{2.353}$$

on the trajectories of motion of the system (2.60), must satisfy the solution of the main functional Bellman equation in partial derivatives

$$\min_U \left[\sum_{i=1}^{n} \frac{\partial S}{\partial \eta_i} \frac{d\eta_i}{dt} + F \right] = 0, \tag{2.354}$$

where S is the Bellman function, which, when synthesizing optimal controls based on the minimum quality functional (2.80), can be uniquely replaced by the Lyapunov function, since the equation (2.81) satisfies the conditions of Lyapunov's stability theorem.

The Lyapunov function $V(\eta)$ for a system of linear differential equations (2.60) is a positive-definite quadratic form

$$V(\eta) = \sum_{i,k=1}^{n} v_{ik}\eta_i\eta_k , v_{ik} = v_{ki} , \tag{2.355}$$

or in matrix form

$$V(\eta) = \mathbf{\eta}^T \mathbf{V} \mathbf{\eta}, \tag{2.356}$$

whose coefficients satisfy the Sylvester criterion.

By differentiating the main functional Bellman equation (2.81) with respect to U, the optimal control that delivers a minimum of the functional (2.80) on the trajectories of the system (2.60) is determined:

$$U = -\sum_{i=1}^{n} \frac{m_i}{2c} \frac{\partial V}{\partial \eta_i}. \qquad (2.357)$$

Matrix **V** is a stationary solution of the Riccati differential equation

$$\mathbf{V}(t) = \mathbf{V}(t)\mathbf{B} + \mathbf{B}^T\mathbf{V}(t) - \mathbf{c}^{-1}\mathbf{V}(t)\mathbf{m}\mathbf{m}^T\mathbf{V}(t) + \mathbf{W},\ \mathbf{V}(\mathbf{0}) = \mathbf{0}. \qquad (2.358)$$

Among all solutions to the Riccati equation (2.85), those that satisfy the Sylvester criterion must be selected, because only in this case is the A.M. Lyapunov theorem on asymptotic stability fulfilled. Determining optimal control in this case is associated with the need to solve the matrix differential equation (2.85), which is a rather complex mathematical problem. One of the ways to simplify the procedure for finding optimal control is to move from a non-stationary matrix $\mathbf{V}(t)$ to a stationary one $\mathbf{V}$. In this case, the coefficients of matrix $\mathbf{V}$ are determined as a result of the numerical solution of the matrix Riccati equation

$$\mathbf{VB} + \mathbf{B}^T\mathbf{V} - \mathbf{c}^{-1}\mathbf{V}\mathbf{m}\mathbf{m}^T\mathbf{V} + \mathbf{W} = \mathbf{0}. \qquad (2.359)$$

A matrix equation (2.86) represents a system $\frac{n(n+1)}{2}$ of nonlinear algebraic equations concerning the same number of unknown coefficients v_{ik} of the Lyapunov function (2.82). If as a result of solving the equation (2.86) this function is found, then the optimal control taking into account the constraint (2.16) takes the final form

$$U = -\mathrm{sat}\left(\sum_{k=1}^{n} \frac{m_i}{c} v_{ik} \eta_k \right), (i = 1, ..., n). \qquad (2.360)$$

As follows from the analysis of the above solution methods for the ADC problem, this problem can only be solved numerically and does not have a strict analytical solution in general form, which makes it difficult to analyze the general properties of synthesized systems.

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Simplifying the computational procedures when solving the ADC problem was proposed by A. A. Krasovsky [28]. To do this, an additional term is introduced into the functional (2.80), taking into account which it takes the expanded form

$$I = \int_0^\infty \left\{ \sum_{i,j=1}^{n} w_{ij}\eta_i\eta_j + cU^2 + \frac{1}{4}\left[\sum_{i=1}^{n} m_i \frac{\partial V(\eta)}{\partial \eta_i}\right]^2 \right\} dt. \qquad (2.361)$$

In a functional (2.88) the function $V(\eta)$ is a quadratic form (2.82) and its coefficients are determined by solving a matrix algebraic equation

$$\mathbf{VB} + \mathbf{B}^T\mathbf{V} + \mathbf{W} = \mathbf{0}, \qquad (2.362)$$

where $\mathbf{B}$ is the matrix of coefficients b_{ik} of the system (2.60) of $n \times n$ dimension, $\mathbf{W}$ is the square matrix of weighting coefficients of the integrand (2.88) of the same dimension.

The equation (2.89) has a unique solution $V > 0$, in particular, when the eigenvalues of the matrix $\mathbf{B}$ have negative real parts, i.e. only when the controlled object is inherently stable. In this case, the synthesized system will be asymptotically stable, and the function $V(\eta)$ will be a Lyapunov function. Then the optimal control algorithm is defined as (2.84). When determining the Lyapunov function satisfying the equation (2.89), it is convenient to use the method proposed by E. A. Barbashin [29]. According to this method, the desired Lyapunov function has the form of a matrix equation

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$$V(\eta)=-\frac{1}{\Delta}\begin{vmatrix} 0 & \eta_1^2 & \cdots & 2\eta_1\eta_n & \eta_2^2 & \cdots & 2\eta_i\eta_k & \cdots & \eta_n^2 \\ -0{,}5w_{11} & c_{11,11} & \cdots & c_{1n,11} & c_{22,11} & \cdots & c_{ik,11} & \cdots & c_{nn,11} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ w_{1n} & c_{11,1n} & \cdots & c_{1n,1n} & c_{22,1n} & \cdots & c_{ik,1n} & \cdots & c_{nn,1n} \\ -0{,}5w_{22} & c_{11,22} & \cdots & c_{1n,22} & c_{22,22} & \cdots & c_{ik,22} & \cdots & c_{nn,22} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -w_{ik} & c_{11,ik} & \cdots & c_{1n,ik} & c_{22,ik} & \cdots & c_{ik,ik} & \cdots & c_{nn,ik} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -0{,}5w_{nn} & c_{11,nn} & \cdots & c_{1n,nn} & c_{22,nn} & \cdots & c_{ik,nn} & \cdots & c_{nn,nn} \end{vmatrix}$$

(2.363)

where $c_{ij,kl}$ are the coefficients that are expressed through the real coefficients of the differential equations of the perturbed motion of the control object (2.60) and are subject to the relations:

$$c_{ij,kl}=c_{ji,kl}=c_{ji,lk};$$

$$c_{ij,kl}=\begin{cases} 0 & \text{if} \quad i\neq j\neq k\neq l; \\ b_{ik} & \text{if} \quad j=l;\, i\neq k; \\ b_{ii}+b_{kk} & \text{if} \quad i=k;\, j=l;\, i\neq j; \\ b_{ii} & \text{if} \quad i=j=k=l, \end{cases} \qquad (2.364)$$

Δ is the minor, related to the element of the first row and first column of the determinant (2.90).

From the equation (2.90) the coefficients of the Lyapunov function v_{ik} are determined using Cramer's formulas

$$v_{ik}=\frac{\Delta_{ik}}{\Delta}, \qquad (2.365)$$

where Δ_{ik} are the algebraic complements of the elements of the first row of the same determinant containing the products $\eta_i\eta_k$.

This solution allows us to express the coefficients v_{ik} explicitly through the coefficients of the differential equations of perturbed motion (2.60) b_{ik} and weight coefficients w_{ij} of functional (2.88). As a result, the control algorithm of the type (2.84) is expressed in analytical form through the parameters of the control object and the quality functional. The functional (2.88) is called the generalized operation criterion, since the last term of the functional (2.88) represents the energy or generalized operation of optimal control U .

All solutions to the ADC problem considered above lead to linear controllers with rigid feedbacks along the coordinates of the perturbed motion of the control object. The control systems synthesized in this way are inherently static and physically incapable to ensure the fulfillment of boundary conditions at the ends of the phase trajectories $\eta(\infty) = 0$, i.e. do not guarantee the convergence of integral quality functionals at $t \to \infty$. In other words, the asymptotic stability of closed-loop system s is not ensured even in the absence of coordinate disturbances. The natural presence of external disturbances exacerbates the situation, because the system acquires a static error not only for the reference input but also for the disturbances. All attempts to take into account external disturbances when solving the ADC problem inevitably lead to a combined control principle, which significantly complicates the control system, but does not provide a fundamental solution to the problem due to the practical impossibility of taking into account and directly measuring the entire spectrum of disturbing influences acting on the object.

Alongside these negative features, the structural property of control systems synthesized as a result of solving ADC problems, such as stability with unlimited increases in the regulator gain, deserves close attention. The implementation of infinitely large gain factors in the domain of linear structures is associated with the need to have an energy source of unlimited power, which is physically unrealizable. However, in the works [30] and [31], it was shown that infinite gain factor can be implemented in relay systems operating in sliding mode. In this regard, the remark of A. M. Letov is significant. He indicated that the choice of a sufficiently small weight coefficient c in functionals of the form (2.61) in the presence of a constraint (2.16) allows one to approach the relay characteristic of the controller as closely as desired, implementing the optimal control law [25]. Indeed, when $c = 0$ the control law (2.87) takes the form

$$U = -\mathrm{sign}\left[\sum_{k=1}^{n} m_i v_{ki} \eta_k\right], (i = 1, \cdots, n). \tag{2.366}$$

The ADC problem for a relay system was solved for the first time by the dynamic programming method based on minimizing the functional

$$I = \int_0^{\infty} \sum_{i,k=1}^{n} w_{ik} \eta_i \eta_k dt \tag{2.367}$$

on the system's motion trajectories (2.60). The optimal control is obtained in the form

$$U = -\mathrm{sign}\left(\sum_{i=1}^{n} m_i \frac{\partial V}{\partial \eta_i}\right), \tag{2.368}$$

where V is the Lyapunov function (2.82), whose coefficients for the system (2.60) and the functional (2.94) are proposed to be determined as a result of solving Bellman's fundamental functional equation after substituting the control input (2.93) into it

$$\sum_{i=1}^{n} \frac{\partial V}{\partial \eta_i} \sum_{k=1}^{n} b_{ik} \eta_k - \left|\sum_{i=1}^{n} m_i \frac{\partial V}{\partial \eta_i}\right| = -\sum_{i,k=1}^{n} w_{ik} \eta_i \eta_k. \tag{2.369}$$

A rigorous solution of a nonlinear partial differential equation (2.96) is complicated by the presence of a sub-signature expression of the control input (2.95), standing under the modulus sign. The existing literature lacks data on the methodology for rigorously solving the equation (2.96). The solution of the equation (2.96) can be carried out based on the condition of the existence of a stable sliding mode in the synthesized closed-loop system, one of the conditions for the occurrence of which is the equality of the average value of the signal at the input of the relay controller to zero, i.e. fulfillment of the relation

$$\left|\sum_{i=1}^{n} m_i \frac{\partial V}{\partial \eta_i}\right| = 0. \tag{2.370}$$

Considering (2.97), the equation (2.96) takes the form

$$\sum_{i=1}^{n} \frac{\partial V}{\partial \eta_i} \sum_{k=1}^{n} b_{ik}\eta_k = -\sum_{i,k=1}^{n} w_{ik}\eta_i\eta_k. \tag{2.371}$$

The expression (2.98) is a system $\frac{n(n+1)}{2}$ of linear algebraic equations and is a particular case of the Riccati matrix equation (2.86). As mentioned above, the solution to (2.86) is the Lyapunov function (2.90).

The given solution to the ADC problem for the case of a relay system was obtained without considering of boundary conditions. The work [32] provides a more rigorous solution to the problem of analytical design of relay controllers by introducing additional restrictions on the expenditures of control signals. In this formulation, the problem was named the analytical design of relay controllers based on the criterion of generalized operation. An additional term is introduced into the functional (2.94), representing a restriction on the "expenditure of control signals"

$$\sum_{i=1}^{n} u_{yim} \int_{t_1}^{t_2} \left|\sum_{k=1}^{n} v_{ik}\eta_k\right| dt = c\left[\eta_1(t), \eta_2(t), \cdots, \eta_n(t)\right], \tag{2.372}$$

where

$$c \le \frac{1}{2} \sum_{i,k=1}^{n} v_{ik}(t_2)\eta_i(t_2)\eta_k(t_2). \tag{2.373}$$

Taking into account (2.99), the quality functional (2.94) takes the form

$$I = \int_{t_1}^{t_2} \sum_{i,k=1}^{n} w_{ik}\eta_i\eta_k dt + \sum_{i,k=1}^{n} v_{ik}(t_2)\eta_i(t_2)\eta_k(t_2). \tag{2.374}$$

In (2.99) u_{yim} is an absolute values of restrictions on control inputs resulting from the condition (2.16)

$$|U(t)| = \left| \frac{u_y(t)}{u_{ym}(t)} \right| \le 1.$$

In the expression (2.99) the quantity

$$\int_{t_1}^{t_2} \left| \sum_{k=1}^{n} v_{ik} \eta_k \right| dt \qquad (2.375)$$

represents an integral estimate of the expenditure of control signals over a time interval $t_1 - t_2$. If $t_1 = 0$ and $t_2 = \infty$, then the integral (2.102) estimates the control signal expenditures over the transient process. With a control input u_{yim} that is significantly limited in modulus, a significant expenditure of the control signal is expected. If such an expenditure rate is unacceptable, it is necessary to ensure a higher level of control input limitation.

If the transient process is limited by the time interval $t_1 - t_2$, then the optimal control problem becomes a problem with fixed ends under terminal control. However, terminal control is often unnecessary and, in this case, the upper time limit remains free. In this case $t_2 \to \infty$, the second term of the functional (2.101) vanishes in accordance with the requirement of asymptotic stability of a closed-loop system $\eta(\infty) = 0$, and the functional (2.101) is transformed to the form (2.94).

It should be emphasized that if not all eigenvalues of a matrix $\mathbf{B}$ have negative real parts and even if there is at least one zero root, i.e. it is not possible to synthesize optimal controls according to the criteria of generalized operation (2.88) and (2.101) by solving the equation (2.89) or (2.90) since in this case the principal determinant Δ of the equation (2.90) has at least one zero column. This is due to the fact that the algorithms for optimal control according to the criteria of generalized operation are calculated based on the Lyapunov function for an open-loop control object and the closing of the system with feedback loops is not taken into account. In addition, arbitrary assignment of weight coefficients w_{ij} in functionals (2.88) and (2.101) may in some cases lead to the synthesis of controls that do not ensure the stability of the corresponding closed-loop system s. In the theory of optimal control, it is traditionally assumed that the optimality criteria are given a priori, leaving the choice of the type of optimizing functional and its coefficients outside the scope of this theory. The controls obtained as a result of solving the ADC problem are optimal only in the sense of the minimum of the assigned functional. In this case, the dynamic properties of the synthesized system may not correspond to the desired ones. In this regard, the selection of such integrands of optimizing functionals, in which optimal systems would have completely defined predetermined properties, is highly relevant.

2.2. The principle of symmetry of self-propelled guns and its modification

The problem of synthesizing automatic control systems (ACS) with the required dynamic properties is, to some extent, connected with the concepts of inverse dynamics problems. The solutions to these problems according to a given law of motion of the system determine the forces or control inputs required to achieve a predetermined motion of the system. In the broadest sense, inverse dynamics problems include the determination of the laws of motion control of dynamic systems and their parameters to ensure that specified trajectories are followed.

Determining the laws of motion control of dynamic systems forms the basis of the structural and algorithmic synthesis of ACS. On the other hand, determining the parameters of a dynamic system is a task of parametric synthesis, where the structure of the control system is assumed to be known a priori. Both of these problems form the basis of the automatic control theory. Despite the long and rich history of inverse dynamics problems, they have gained increasing recognition and understanding in recent years, particularly within the field of automatic control theory (ACT). The essence of these problems is to construct closed-loop ACS that move along designated trajectories (trajectories of unperturbed motion) using control laws with feedback based on the state variables of the control objects.

In the process of development of classical and modern ACT, regardless of the progress made in solving inverse dynamics problems, many practical techniques and methods for creating and calculating ACS of various natures and purposes have been developed. These primarily include frequency and root methods, as well as the problems of analytical design of optimal controllers by minimizing integral quality functionals, as discussed earlier. In any formulation of the synthesis problem using these methods, the ultimate goal is to determine the structures and parameters of the ACS that ensure the dynamic processes of the system follow prescribed laws or closely approximate the processes occurring in a certain reference model, which best meets the technical design requirements. A similar goal is pursued by solutions of inverse dynamics problems. This allows us to conclude that the synthesis methods used in ACT are directly or indirectly connected to the concepts of inverse dynamics problems. In each of these methods, either explicitly or implicitly, a reference model of unperturbed motion is defined, and the task is to determine the control law as a function of the state variables of the controlled object, ensuring the system follows the specified trajectories.

2.2.1. Concept of inverse problems of dynamics

Let us consider the classical problem of automatic control theory, which involves determining the structure and parameters of a control law [33]. Using the known transfer function of control object $W(p)$, it the task is to find the transfer function of the control device $W_y(p)$ such that the closed-loop automatic control system has the desired transfer function $\Phi^*(p)$. In this formulation of the problem, the desired motion trajectory or the required dynamic properties of the synthesized system are specified by the type of transfer function $\Phi^*(p)$. The required control law is represented by the transfer function of the control device $W_y(p)$. The combination of the control object and the control device forms a closed-loop ACS. The determination of the control law and its parameters in the form of a transfer function $W_y(p)$ based on the available initial data corresponds to the content of inverse dynamics problems. The block diagram of the synthesized system is shown in Fig. 2.5.

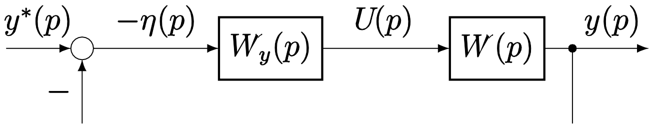

Fig. 2.5. Generalized block diagram

Transfer function of the closed-loop system shown in Fig. 2.5 has the following form

$$\Phi(p) = \frac{W_y(p)W(p)}{1 + W_y(p)W(p)}. \quad (2.376)$$

Substituting the desired transfer function $\Phi^*(p)$ of the closed-loop system instead of the transfer function $\Phi(p)$ and solving the resulting equation for $W_y(p)$, we find the desired control law in the form of a transfer function

$$W_y(p) = \frac{\Phi^*(p)}{W(p)(1 - \Phi^*(p))}. \quad (2.377)$$

The expression (2.104) formally solves the problem of synthesizing a closed-loop system under the condition of implementing a given transfer function. This indicates that the traditional problem of automatic control theory is formulated and solved as an inverse dynamics problem in its direct understanding. The desired motion trajectory of the synthesized system is specified in the form of a transfer function $\Phi^*(p)$ of certain reference model, and the desired control law is also defined in the form of a transfer function $W_y(p)$ or equation $U(p) = W_y(p)[y^*(p) - y(p)]$.

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Let us consider the synthesis of a control system that ensures the movement of a representing point along a specified trajectory as a solution to the inverse dynamics problem in a different formulation. Let the motion of the control object to be governed by differential equations

$$py_i = \sum_{k=1}^{n} b_{ik} y_k + m_n u, \; (i = 1, \dots, n). \tag{2.378}$$

The system of n equations (2.105) can be reduced to a single differential equation of n-th order

$$\sum_{k=0}^{n} a_k p^k y_1 = m_n u. \tag{2.379}$$

The task is to determine the control input u that will ensure the movement of the coordinate $y_1(t)$ along the trajectory $y_1^*(t)$.

In accordance with the basic idea of inverse dynamics problems, we determine the control input from the equation (2.106)

$$u = m_n^{-1} \sum_{k=0}^{n} a_k p^k y_1(t). \tag{2.380}$$

Substituting the desired value $y_1^*(t)$ instead of the current value of the variable $y_1(t)$ in (2.107) we will get

$$u^* = \frac{1}{m_n} \sum_{k=0}^{n} a_k p^k y_1^*(t). \tag{2.381}$$

From the expression (2.108), it follows that the desired control input can be found as a function of time by performing a finite number of operations: differentiation, addition, multiplication, etc.

Based on the relationship (2.108), general provisions for determining the control inputs that ensure the system’s movement along the designated trajectory are formulated. From the comparison of expressions (2.106) ant (2.108), it follows that the operations of forming the desired control are the inverse of the corresponding operations that determine the structure of the mathematical model of the control object. Integration in the mathematical model of an object corresponds to differentiation in the control algorithm, summation corresponds to subtraction, and multiplication corresponds to division. Ultimately, the output u^* and input y_1^* variables of the control algorithm block diagram represent the corresponding inverted variables u, y_1 of the mathematical model of the controlled object. Thus, the block diagram of the control part of the system can be obtained based on the block diagram of the control object by reversing the operations and corresponding variables.

Fig. 2.6 shows a block diagram of the control system, built in accordance with the equations (2.106) and (2.107) at $n = 2$.

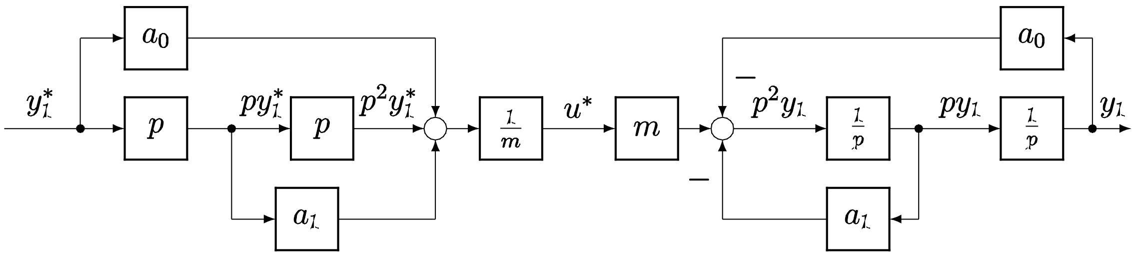

Fig. 2.6. System block diagram

The input variable of the diagram is the trajectory of unperturbed motion y_1^*, and the output variable is the actual variable y_1. If the corresponding operations and variables are reversed, the direction of the diagram will change, i.e. the output of the system will become its input and vice versa, and the overall configuration of the block diagram will not change. As a result of such a reversal, the structural diagram will take the form shown in Fig. 2.7.

Thus, the following rule can be formulated: the algorithm for forming a control input is built on the principle of structural symmetry and reversing operations concerning the structure and group of operations corresponding to the mathematical model of the controlled process [33].

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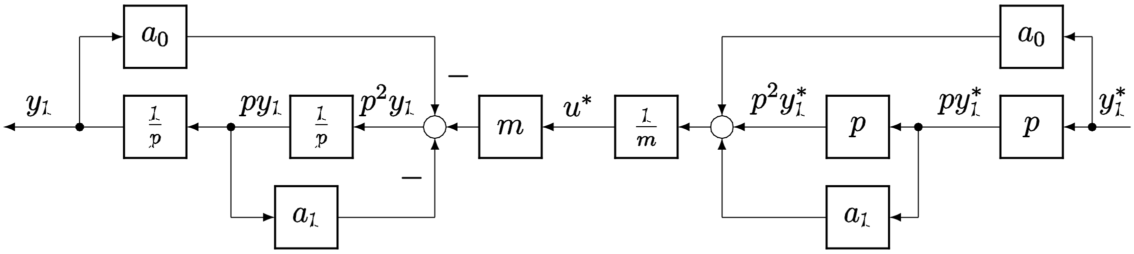

Fig. 2.7. Block diagram of the inverted system

2.2.2. Symmetry property of automatic control systems

The given block diagrams clearly illustrate the symmetry property inherent in automatic control systems. These properties unambiguously determine the structure and parameters of the control part of the system and form the methodological basis for finding control algorithms for the movement of dynamic objects along a designated trajectory. In this case, the task of designing control algorithms fully aligns with the concepts of inverse dynamics problems and comes down to finding a control function u^* that ensures the movement of an object along a designated trajectory y_i^*. In the considered formulation, the solution to the inverse dynamics problem makes it possible to determine program controls that provide the system with predetermined dynamic properties, if the designated trajectory of motion is given by the equation

$$py_1^*(t) = \sum_{i=1}^{n} c_i x_i(t), \qquad (2.382)$$

where $x_i(t)$ are known functions of time; c_i are constants that uniquely determine the initial state of the system.

By substituting the expression (2.109) into (2.108) we determine the required program control

$$u^*(t) = \frac{1}{m}\left[c_1 \sum_{k=0}^{n} a_k x_1^{(k)}(t) + \cdots + c_n \sum_{k=0}^{n} a_k x_n^{(k)}(t) \right]. \qquad (2.383)$$

It is significant that controls of the form (2.110) ensure the implementation of

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only those motion trajectories, whose structure as functions of time corresponds to the structure of the solution to the equation (2.106) for $u = 0$. It follows from this that program controls are implemented only if the conditions for the reproducibility of the designated trajectories are met.

It is more preferable to implement designated motion trajectories in closed-loop systems based on control laws with feedback. To find such laws, it is necessary to express the time functions $c_i x_i^{(k)}(t)$ in terms of the state variables of the control object. This problem is easily solved for functions $x_i(t)$ that satisfy the relation

$$p^k x_i(t) = \sum_{i=1}^{n} \rho_{ik} x_i(t), (k = 0, l, ..., n). \tag{2.384}$$

If $x_i(t)$ in the expression of the designated trajectory (2.109) meets the condition (2.111), then the program control (2.110) takes the form

$$u^*(t) = \frac{1}{m} \sum_{i=1}^{n} \gamma_i c_i x_i(t), \tag{2.385}$$

where the coefficients γ_i are determined by the expressions

$$\gamma_i = \sum_{k=0}^{n} a_k \rho_{ik}, (i = 1, ..., n). \tag{2.386}$$

To construct a feedback control law based on the program control (2.112), it is necessary to express functions $c_i x_i(t)$ in terms of system state variables (2.106). Such variables are $p^k y_i(t), (k = 0, 1, ..., n-1)$. Then, in accordance with the condition of reproducibility of the designated motion trajectory $y_i(t) = y_i^*(t)$, the following expression can be obtained

$$\sum_{i=1}^{n} c_i \rho_{ik} x_i(t) = y^{(k)}(t), (k = 0, 1, \dots n-1). \tag{2.387}$$

By solving the system of equations (2.114) with respect to the desired variables

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$c_i x_i(t)$, we determine

$$c_i x_i(t) = \sum_{k=0}^{n-1} \beta_{ik} \rho^k y_1(t), (i = 1, \cdots, n), \tag{2.388}$$

where β_{ik} are constant coefficients.

Substituting expressions (2.115) into (2.112), we obtain the feedback control law

$$u*(y_1) = m^{-1} \sum_{k=0}^{n-1} \delta_k p^k y_1. \tag{2.389}$$

The trajectory of unperturbed motion $y^*(t)$, specified by the equation (2.109), is implemented in a closed-loop system with a control algorithm (2.116) in which the feedback coefficients for the variables $y_i, py_i, \ldots, p^{n-1} y_i$ are determined by the expression

$$\delta_k = \sum_{i=1}^{n} \gamma_i \beta_{ik}, (k = 0, 1, \ldots, n-1). \tag{2.390}$$

Thus, as a result of solving the inverse dynamics problem based on the principle of symmetry, a control algorithm for a closed-loop system is obtained in analytical form as a function of state variables and parameters of the control object, as well as known time functions that determine the type of the designated motion trajectory, taking into account the initial state of the system.

2.2.3. Modification of the principle of symmetry

The solution of inverse dynamics problems is based on the assumption that the desired motion trajectory is known in advance and specified in some form. Methods for defining such trajectories inevitably contain elements of subjectivity and do not exclude the possibility of using iterative procedures in the process of finding optimal solutions that best satisfy the complex of conflicting requirements of technical specifications for the design of control systems. The situation is further complicated by the fact that the presence of natural restrictions on the maximum values of control inputs and certain state variables of the control object requires the formation of system motion trajectories, consisting of segments of phase trajectories corresponding to the solutions of the dynamics equations, meeting these restrictions. This is particularly relevant to control systems, the synthesis of which places high demands not only on the accuracy of stabilization of steady-state movements, but also on speed in transient modes. In the vast majority of cases, the driving inputs for automated electric drive control systems are step functions, and the implementation of control laws obtained from solving inverse dynamics problems inevitably leads to the need to construct trajectory generators for unperturbed motion, which is not always economically and technically justified.

In this regard, the problem of developing new effective methods for structural-algorithmic synthesis of control systems that best satisfy a set of conflicting engineering requirements for control quality under the influence of a wide range of destabilizing factors is highly relevant. Another important requirement is the simplicity of computational procedures during the determination of control algorithms and fulfillment of the conditions for their technical feasibility.

Below in this chapter we will study the tasks of designing control algorithms for the motion of objects whose dynamics are described by linear or linearized differential equations, based on general provisions determined by the symmetry properties and solutions to problems of analytical design of controllers. The use of these provisions made it possible to develop an effective procedures for the synthesis of control laws, ensuring the maximum dynamic characteristics of the controlled process under restrictions on control inputs, while taking into account the requirements for the static properties of the synthesized structures.

Let us consider the block diagram shown in Fig. 2.6. Moving on to the apparatus of transfer functions, this scheme can be represented in the form shown in Fig. 2.8

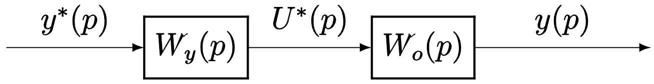

Fig. 2.8. Open-loop system built based on the symmetry principle

It is easy to see that for a linear control object with a transfer function of the general form $W_0(p) = \dfrac{M(p)}{N(p)}$, where $M(p) = \sum_{i=0}^{m} b_i p^i$, $N(p) = \sum_{i=0}^{n} a_i p^i$ are polynomials of degree m and n ($m \le n$) respectively; b_i and a_i are real numbers, the transfer function of the control part has the form

$$W_y(p) = \frac{N(p)}{M(p)} = \frac{1}{W_0(p)}. \qquad (2.391)$$

In other words, the transfer function of the control part of the system is the inverse of the transfer function of the controlled object. As a result, the equivalent transfer function of the considered open-loop system equals one, which determines the condition for ideal reproduction of a given motion trajectory $y_1 = y_1^*$. This condition is realizable only if the zeros and poles of the transfer functions of the controlled object $W_o(p)$ and the control device $W_y(p)$ cancel each other out. It follows that the reproduction of given motion trajectories in open-loop systems built on the basis of the symmetry principle is possible only for stationary objects with frozen parameters in the absence of external disturbances. For real electromechanical objects, the considered control principle in an open-loop system is meaningless.

Let us close the system by including an amplifying element with a transmission coefficient g. As a result, we obtain the block diagram shown in Fig. 2.9.

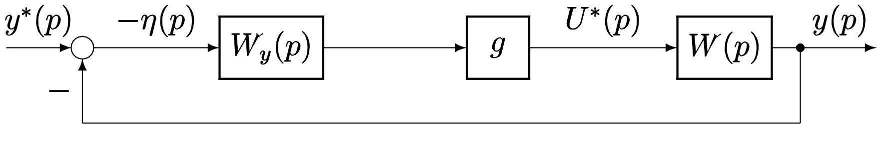

Fig. 2.9. To the design of a closed-loop system block diagram.

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The transfer function of such system is

$$\Phi(p)=\frac{y(p)}{y^{*}(p)}=\frac{gW_{y}(p)W_{0}(p)}{1+gW_{y}(p)W_{0}(p)} \tag{2.392}$$

if the condition (2.118) is met, then it will take the form

$$\Phi=\frac{y(p)}{y^{*}(p)}=\frac{g}{1+g}. \tag{2.393}$$

The expression (2.120) indicates that the use of symmetry properties theoretically makes it possible to construct an inertialess closed-loop control system, and condition that $g \to \infty$ ensures that the transfer function of such a system equals one. Consequently, the control part of the system must fully compensate for the inherent dynamics of the control object and ensure ideal reproduction of the stepwise reference input. In real electromechanical objects, this is only possible if there is an energy source of unlimited power, which is physically impossible. In addition, in existing electric drive control systems there is always a limitation on the maximum value of the control input. Taking this into account, we modify the closed-loop system, the block diagram of which is shown in Fig. 2.9, by introducing an integrating element into its composition, which contributes to the formation of realistically achievable maximum dynamic characteristics in a closed state and, if necessary, ensures the elimination of a steady-state error in the driving force with a limited gain factor. In addition, we will take into account the limitation of the maximum control input value by including a saturation function sat in the appropriate location of the direct amplification path of the system. As a result, we obtain a system, the block diagram of which is shown in Fig. 2.10.

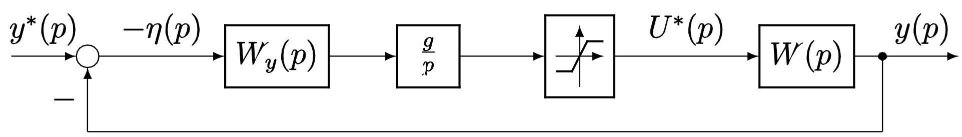

Fig. 2.10. System with an integral component

In such a system, the control input $u(p)$ directly arriving at the input of the control object equals to the optimal control $u^*(p)$ if $\left|u^*(p)\right| \le 1$ and $u(p) = \pm 1$ otherwise.

Let us consider the transfer function of a closed-loop system when $u(p) = u^*(p)$, i.e. when the control input at the object input has not reached the saturation level

$$\Phi(p) = \frac{y(p)}{y^*(p)} = \frac{\frac{g}{p} W_y(p) W_o(p)}{1 + \frac{g}{p} W_y(p) W_o(p)}. \qquad (2.394)$$

If the condition (2.118) is met, the transfer function (2.121) is transformed into

$$\Phi(p) = \frac{y(p)}{y^*(p)} = \frac{1}{\frac{1}{g} p + 1}. \qquad (2.395)$$

In other words, a closed-loop control system is equivalent to a first-order aperiodic element with a time constant $T = 1/g$. When a stepwise reference input y^* of a sufficiently large amplitude is applied to the input of the system, its movement will occur along two segments of phase trajectories. Initially, under the influence of a significant in absolute value input y_1^*, the controller will saturate and the control system will operate as an open-loop with the maximum permissible control input $u = 1$. The motion trajectory of the output variable y is determined exclusively by the dynamic parameters of the control object and the level of control input saturation. As the system accelerates, the error η will begin to decrease until u^* becomes equal to one. At this moment, the system will close and further movement will occur exponentially $\exp\left(\frac{1}{g} t\right)$ in accordance with the transfer function (2.122) until the error

η_0 becomes zero. Thus, by choosing the controller gain g, it is possible to ensure that the transition process of bringing the system from an arbitrary initial position η_0 to the origin $\eta = 0$ at the final stage occurs with the required intensity. If the control input $y^*(t)$ as a function of time is a reproducible trajectory, during the execution of which in any section the control input $u^*(t)$ does not reach the saturation level, then the system remains closed, and behaves as an astatic first-order system. The ideal reproduction of such unperturbed movements with zero error can be achieved in the sliding mode, when $g = \infty$, and the function sat is transformed into a function sign.

2.2.4. Correspondence of the modified symmetry principle to solutions to the problem of analytical design of regulators

Such an approach to the synthesis of control algorithms under certain conditions fully corresponds to solutions to the problem of analytical design of controllers. Let us demonstrate this using the example of a control object of arbitrary order, the movement of which is described by a normal system of differential equations in the Cauchy form

$$py_i = \sum_{k=1}^{n} b_{ik} y_k + m_n u, \; (i = 1, 2, \cdots, n), \tag{2.396}$$

where y_k, u are state variables and control inputs in relative units, respectively; b_{ik}, m_n are constant coefficients.

Let us transform the system (2.123) into normal form

$$\begin{aligned} p\hat{y}_i &= \hat{y}_{i+1}; \; i = 1, 2, \cdots, n-1; \\ p\hat{y}_n &= \sum_{i=1}^{n} -a_i \hat{y}_i + M_n u, \end{aligned} \tag{2.397}$$

or in expanded form

$$
\begin{aligned}
&p\hat{y}_1 = \hat{y}_2;\\
&p\hat{y}_2 = \hat{y}_3;\\
&\vdots\\
&p\hat{y}_{n-1} = \hat{y}_n;\\
&p\hat{y}_n = -a_1\hat{y}_1 - a_2\hat{y}_2 - \cdots - a_n\hat{y}_n + M_n u.
\end{aligned}
\quad (2.398)
$$

The transition from system (2.123) to system (2.125) can be achieved using the so-called *linear non-singular transformation*. It is important to note that in system (2.125) $\hat{y}_1, \cdots, \hat{y}_n$ are some fictitious coordinates that have been intentionally introduced into the system, and some of them may coincide with the real phase coordinates of the control object. In this case $\hat{y}_1$ coincides with y_1. Systems (2.123) and (2.125) describe the motion of the same object in different phase spaces.

The last equation of the system (2.125) can be written as

$$p^n\hat{y}_1 = -a_1\hat{y}_1 - a_2 p\hat{y}_1 - a_3 p^2\hat{y}_1 - \cdots - a_n p^{n-1}\hat{y}_1 + M_n u, \quad (2.399)$$

expressing all terms on the left and right sides through the derivatives of the output coordinate, because for the system (2.125)

$$\hat{y}_2 = p\hat{y}_1,\ \hat{y}_3 = p^2\hat{y}_1, ,\cdots,\ \hat{y}_n = p^{n-1}\hat{y}_1,\ p\hat{y}_n = p^n\hat{y}_1. \quad (2.400)$$

The expression (2.126) describes the dynamics of the control object specified by the transfer function of the form

$$W_o(p) = \frac{y_1(p)}{u(p)} = \frac{M_n}{p^n + a_n p^{n-1} + \cdots + a_2 p + a_1}. \quad (2.401)$$

Thus, the coefficients $\mathrm{a}_i, (i = 1, \cdots, n)$ in the last equation of the system (2.125) are the coefficients of the characteristic polynomial of the control object (2.123). In accordance with the modified symmetry principle stated above and the block diagram in Fig. 2.10, let us find the control input

$$u = -\mathrm{sat}\left[\frac{g}{M_n}\left(\frac{a_1}{p} + a_2 + a_3 p + \cdots + a_n p^{n-2} + p^{n-1}\right)\eta_1\right]. \quad (2.402)$$

Now let us determine the optimal control of the object (2.123) as a result of solving the problem of analytical design of controllers, for which we describe its dynamics by equations of perturbed motion corresponding to the system (2.125)

$$\begin{aligned} p\hat{\eta}_i &= \hat{\eta}_{i+1};\, i = 1, 2, \cdots, n-1; \\ p\hat{\eta}_n &= -\sum_{i=1}^{n} a_i \eta_i + M_n U, \end{aligned} \tag{2.403}$$

where $\hat{\eta}_i = \hat{y}_i - \hat{y}_i^*, (i = 1, \cdots, n)$ are the deviations of the coordinates of the true motion of the system (2.125) from the unperturbed one; U is a stabilizing control.

We will assume that the quality of control is specified by the functional

$$I = \int_0^{\infty} \left(\sum_{i,k=1}^{n} \hat{w}_{ik} \hat{\eta}_i \hat{\eta}_k + cU^2 \right) dt, \tag{2.404}$$

and find a control $U(\hat{\eta}_1, \cdots, \hat{\eta}_n)$ that minimizes the integral (2.131) on the motion trajectories of the system (2.130) from any initial position $\hat{\eta}_{10}, \cdots, \hat{\eta}_{n0}$ to the origin $\hat{\eta}_1(\infty) = \cdots = \hat{\eta}_n(\infty) = 0$.

Let us create the basic functional Bellman equation for the system (2.130) and the functional (2.131)

$$\sum_{i=1}^{n} \frac{\partial \hat{V}}{\partial \hat{\eta}_i} p\hat{\eta}_i + \sum_{i,k=1}^{n} \hat{w}_{ik} \hat{\eta}_i \hat{\eta}_k + cU^2 = 0, \tag{2.405}$$

or in expanded form

$$\begin{aligned} &\frac{\partial \hat{V}}{\partial \hat{\eta}_n} \left(-a_1 \hat{\eta}_1 - a_2 \hat{\eta}_2 - \cdots - a_n \hat{\eta}_n + M_n U \right) + \\ &+ \frac{\partial \hat{V}}{\partial \hat{\eta}_1} \hat{\eta}_2 + \frac{\partial \hat{V}}{\partial \hat{\eta}_2} \hat{\eta}_3 + \cdots + \frac{\partial \hat{V}}{\partial \hat{\eta}_{n-1}} \hat{\eta}_n + \sum_{i,k=1}^{n} \hat{w}_{ik} \hat{\eta}_i \hat{\eta}_k + cU^2 = 0. \end{aligned} \tag{2.406}$$

To determine the desired control, we differentiate the expression (2.133) with respect to U

$$\frac{\partial \hat{V}}{\partial \hat{\eta}_n} M_n + 2cU = 0, \quad (2.407)$$

where

$$U = -\frac{M_n}{2c}\frac{\partial \hat{V}}{\partial \hat{\eta}_n}. \quad (2.408)$$

In the algorithm (2.135) $\hat{V}$ is the Lyapunov function, which is defined by the expression

$$\hat{V} = \sum_{i,k=1}^{n} \hat{v}_{ik}\hat{\eta}_i\hat{\eta}_k,\, \hat{v}_{ik} = \hat{v}_{ki}. \quad (2.409)$$

Taking into account the Lyapunov function, (2.136)the optimal control (2.135) takes the form

$$U = -\frac{M_n}{c}\left(\hat{v}_{1n}\hat{\eta}_1 + \hat{v}_{2n}\hat{\eta}_2 + \cdots + \hat{v}_{nn}\hat{\eta}_n\right). \quad (2.410)$$

Taking into account the fact that in real systems there is a limitation of the control input $|U| \le 1$, and expressing the variables $\hat{\eta}_2, \cdots, \hat{\eta}_n$ through $\eta_1 = \hat{\eta}_1$ based on equations (2.130), we finally get

$$U = -sat\left[\frac{M_n}{c}\left(\hat{v}_{1n} + \hat{v}_{2n}p + \cdots \hat{v}_{nn}p^{n-1}\right)\eta_1\right]. \quad (2.411)$$

Comparing the algorithms (2.129) and (2.138), we can see that except for the integral component $\frac{a_1}{p}$ in (2.129), they have the same structure. This conclusion allows us to state that the solution to the ADC problem can be interpreted in terms of the concepts of inverse dynamics problems, arising from the symmetry properties of automatic control systems.

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2.3. Modal control

The results presented in the previous section were obtained based on the symmetry principle in control systems, which connects their structure to the desired trajectories of movement. Achieving these trajectories is impossible without properly assigned zeros and poles of the transfer function of a closed-loop ACS. Therefore, it is of interest to determine the correspondence between the solution of the problem of ADC using the modified symmetry principle to the basic provisions of the modal control theory [34]. However, before moving on to establishing this correspondence, let us consider the basic methods and approaches of the modal control theory.

2.3.1. The concept of modal control

Modal control is a control method based on placing the roots of the characteristic equation in a certain predetermined manner [34]. The desired root distribution is ensured by applying a specially designed linear control input.

Let us prove this statement with the following statements.

Let there be given: a linear time-invariant object

$$p\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \; \mathbf{x} \in R^n, \; u \in R, \tag{2.412}$$

and $l \; (l \leq n/2)$ pairs of arbitrary complex conjugate numbers $p_i = \alpha_i \pm jb_i, \; (i = 1, 2, \ldots, l)$ and $n - 2l$ arbitrary real numbers $p_s = \alpha_s, \; (s = 2l + 1, \ldots, n)$.

If a linear object (2.139) is controllable, then there is a linear control law

$$u = k^T x = k_1 x_1 + k_2 x_2 + \ldots + k_n x_n, \tag{2.413}$$

where the roots of the characteristic equation of the closed-loop control system are equal to the specified numbers.

Using a transformation $\mathbf{x} = \mathbf{T}\mathbf{z}$ that turns the equations of the object into the controlled Luenberger form (2.139), we define the characteristic equation of the object

$$\det(p\mathbf{E}-\mathbf{A}) = p^n + a_1 p^{n-1} + \ldots + a_n = 0. \tag{2.414}$$

In order for the roots of the characteristic equation of a closed-loop system to be equal to the specified numbers, it must have the form

$$\prod_{i=1}^{l}(p-\alpha_i - j\beta_i)(p-\alpha_i + j\beta_i)\prod_{i=1}^{l}(p-\alpha_s) =$$
$$= p^n + c_1 p^{n-1} + \ldots + c_n = 0. \tag{2.415}$$

The characteristic equation of a closed-loop system will have the following form if we accept the control law

$$u = (a_n - c_n)z_1 + (a_{n-1} - c_{n-1})z_2 + \ldots + (a_1 - c_1)z_n =$$
$$= (\mathbf{a}-\mathbf{c})^T \mathbf{z}, \tag{2.416}$$

where $\mathbf{a} = (a_n \quad a_{n-1} \quad \ldots \quad a_1)^T$; $\mathbf{c} = (c_n \quad c_{n-1} \quad \ldots \quad c_1)^T$.

Indeed, substituting this control law into the equations of motion of the object, we obtain

$$p\mathbf{z} = \mathbf{A}\mathbf{z} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -c_n & -c_{n-1} & \ldots & -c_1 \end{bmatrix}\mathbf{z} \tag{2.417}$$

The desired control law can be determined by substituting the expression $\mathbf{z} = \mathbf{T}^{-1}\mathbf{x}$ into the formula (2.143)

$$u = (\mathbf{a}-\mathbf{c})^T \mathbf{z} = (\mathbf{a}-\mathbf{c})^T \mathbf{T}^{-1}\mathbf{x}, \tag{2.418}$$

or

$$u = \mathbf{k}^T \mathbf{x}, \;\; \mathbf{k}^T = (\mathbf{a}-\mathbf{c})^T \mathbf{T}^{-1}. \tag{2.419}$$

A similar statement is true for multivariable control systems: if a linear time-invariant control object $p\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, $\mathbf{x} \in R^n$, $\mathbf{u} \in R^r$ and $l(l \le n/2)$ pairs of arbitrary complex conjugate numbers $p_i = \alpha_i \pm jb_i, \;\; (i = 1,2,\ldots,l)$, as well as

$n-2l$ arbitrary real numbers $p_s = \alpha_s$, $(s = 2l+1, \ldots, n)$ are given, then it is possible to determine the control law $\mathbf{u} = \mathbf{Kx}$, where $\mathbf{K}$ is the matrix of $(r \times n)$ dimension, in which the roots of the characteristic equation of the closed-loop system are equal to the specified numbers.

The above statements can be generalized and used to determine a controller algorithm that guarantees the desired root distribution. Let us consider a generalized procedure for the synthesis of such a controller.

Let the characteristic equation of a one-dimensional control system

$$p\mathbf{x} = \mathbf{Ax} + \mathbf{B}u;\ \mathbf{x} \in R^n,\ u \in R, \tag{2.420}$$

to look like

$$p^n + a_1 p^{n-1} + \ldots + a_n = 0. \tag{2.421}$$

In order for the characteristic equation of a closed-loop system to correspond to the desired

$$p^n + c_1 p^{n-1} + \ldots + c_n = 0, \tag{2.422}$$

it is necessary to select a control law of the form

$$u = (\mathbf{a} - \mathbf{c})^T \mathbf{T}^{-1} \mathbf{x}, \tag{2.423}$$

where

$$\mathbf{a} = (a_n \quad a_{n-1} \quad \ldots \quad a_1)^T;\ \mathbf{c} = (c_n \quad c_{n-1} \quad \ldots \quad c_1)^T;$$

$$\mathbf{T}^{-1} = \left(h^T \quad h^T A \quad \ldots \quad h^T A^{n-1}\right)^T,$$

the vector variable $\mathbf{h}^T$ in the transformation matrix $\mathbf{T}$ is determined by solving the following equations

$$\begin{aligned} &\mathbf{h}^T \mathbf{B} = 0; \\ &\mathbf{h}^T \mathbf{AB} = 0; \\ &\ldots \\ &\mathbf{h}^T \mathbf{A}^{n-1} \mathbf{B} = 0. \end{aligned} \tag{2.424}$$

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2.3.2. Standard distributions of the roots of the characteristic equation

A significant disadvantage of modal control is that the desired roots of the characteristic equation are considered known or somehow specified. For example, they can be obtained as a result of empirical research of the control object. In general, the desired roots of the characteristic equation can be arbitrary, which significantly complicates the synthesis of a control system. The creation of such a system should be preceded by determining the influence of the roots on the dynamic and static properties of the control system. To avoid conducting these studies when synthesizing each modal controller, several standard polynomials were proposed [34], whose roots follow certain relations and are located in a specific way on the complex plane of the characteristic polynomial's roots of the system. The properties and characteristics of control systems whose characteristic equations correspond to standard polynomials have been well studied. The most common distributions of the roots of the characteristic equation and the corresponding polynomials are given in Table 2.1.

Table 2.1. Standard distributions of the characteristic equation roots

n	Distribution name	Polynomial name
1	Newton distribution	Binomial polynomial
2	Butterworth distribution	Butterworth polynomial
3	Distribution that provides the minimum integral of the squared control error	Corresponding polynomial
4	Distribution that provides the minimum integral of the product of the control error and the control input	Corresponding polynomial

In addition to the distributions listed, Bessel and Chebyshev distributions are sometimes used, implemented with the corresponding polynomials. Other standard distributions of the characteristic equation roots may also be used.

Let us consider the distributions presented in table 2.1 in more detail.

All standard polynomials depend on the parameter ω_0 [6]. This parameter determines the radius of root distribution of the characteristic polynomial. Therefore,

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this value is always positive, that is $\omega_0 > 0$. It should be noted that as the parameter ω_0 increases, the control time decreases, but at the same time the control input supplied to the object increases and therefore, starting from certain values of ω_0, the controller may enter saturation.

One assumption used to justify the choice of standard characteristic polynomials is to ensure uniformity of all roots of the characteristic equation, and the n-fold root of this equation must be a real negative, with a modulus value of ω_0. Then the characteristic equation turns into a Newton's binomial

$$D^*(p) = (p + \omega_0)^n, \tag{2.425}$$

expanding which, we obtain the standard desired values of the coefficients of the characteristic equation.

Newton polynomials from the first to the sixth order are presented in Table 2.2.

Table 2.2. Newton polynomials

n	Standard Newton polynomial
1	$p+\omega_0$
2	$p^2+2\omega_0 p+\omega_0^2$
3	$p^3+3\omega_0^2 p+3\omega_0 p^2+\omega_0^3$
4	$p^4+4\omega_0^3 p+6\omega_0^2 p^2+4\omega_0 p^3+\omega_0^4$
5	$p^5+5\omega_0^4 p+10\omega_0^3 p^2+10\omega_0^2 p^3+5\omega_0 p^4+\omega_0^5$
6	$p^6+6\omega_0^5 p+15\omega_0^4 p^2+20\omega_0^3 p^3+15\omega_0^2 p^4+6\omega_0 p^5+\omega_0^6$

As can be seen from the expression (2.152), the characteristic polynomial has negative real multiple roots, which are equal to:

$$p_i = -\omega_0, \quad i = 1,...,n. \tag{2.426}$$

On the complex plane, such a distribution of roots has the form shown in Fig. 2.11.

The roots of the desired polynomial, being negative real, guarantee the aperiodic nature of transient processes in a closed-loop system, i.e. they allow obtaining transient processes with zero overshoot.

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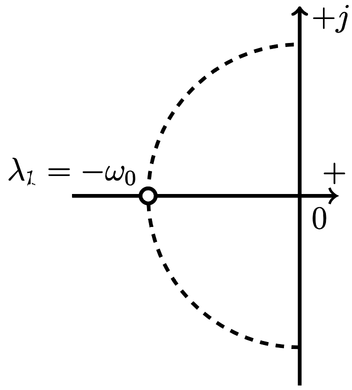

Fig. 2.11. Complex plane of Characteristic Equation Roots

To form the Newton polynomial, it is necessary to know the value of the parameter ω_0, which is determined based on the method of standard transition functions.

This method is based on the normalization of transition functions, which is carried out in time relative to the parameter ω_0. As a result, for any value of ω_0 the transition function is normalized at a fixed polynomial order. The form of the transition function corresponds to a transfer function that has only poles and the ratio of the free coefficients of these polynomials is equal to 1.

Normalization of transition functions is carried out by replacing the value of the parameter ω_0 by one. Graphs of normalized transition functions for control systems whose characteristic equations follow the Newton distribution are presented in Fig. 2.12.

Despite the aperiodic nature of the transient process, such a distribution is unsuitable for many applications because it does not provide the required speed of response.

In the standard Butterworth polynomial, all roots are distributed in the left half-plane of the complex root plane on a semicircle with radius ω_0. Moreover, the angle between adjacent root radius vectors is π / n, and the angle between the closest root radius vector and the imaginary axis is 0.5π /n.

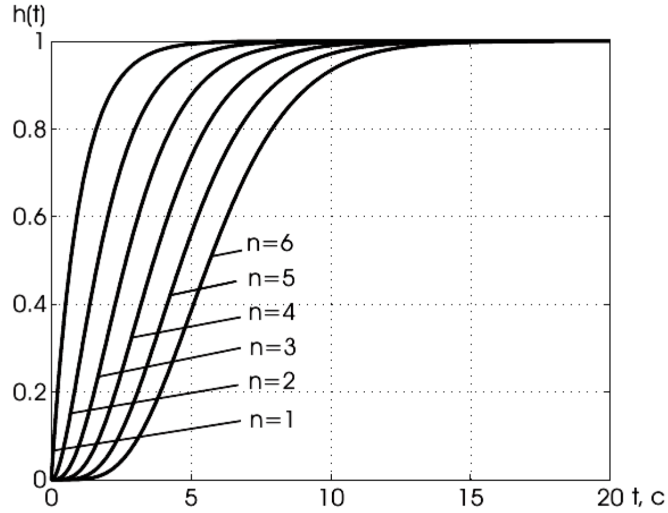

Fig. 2.12. Normalized transition characteristics of Newton polynomial

Therefore, the roots of such a polynomial are found by the formula:

$$p_i^* = \omega_0 \left(\cos\left(\frac{\pi(2i-1)}{n} \right) + j\sin\left(\frac{\pi(2i-1)}{n} \right) \right), i = 1,...,n. \tag{2.427}$$

The Butterworth polynomial is described by the following expression

$$D^*(p) = \prod_{i=1}^{n} (p - p_i^*) . \tag{2.428}$$

Butterworth polynomials from the first to the sixth order are presented in Table 2.3. Table 2.4 provides the root values of the polynomials from the Table 2.3.

Table 2.3. Butterworth polynomials

n	Standard Butterworth polynomial
1	$p + \omega_0$
2	$p^2 + 1,4\omega_0 p + \omega_0^2$
3	$p^3 + 2\omega_0^2 p + 2\omega_0 p^2 + \omega_0^3$
4	$p^4 + 2,6\omega_0^3 p + 3,4\omega_0^2 p^2 + 2,6\omega_0 p^3 + \omega_0^4$
5	$p^5 + 3,24\omega_0^4 p + 5,24\omega_0^3 p^2 + 5,24\omega_0^2 p^3 + 3,24\omega_0 p^4 + \omega_0^5$
6	$p^6 + 3,86\omega_0^5 p + 7,46\omega_0^4 p^2 + 9,13\omega_0^3 p^3 + 7,46\omega_0^2 p^4 + 3,86\omega_0 p^5 + \omega_0^6$

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Table 2.4. Roots of Butterworth polynomials

n	Polynomial Roots
1	$-\omega_0$
2	$(-0{,}71 \pm j0{,}71)\omega_0$
3	$-\omega_0, (-0{,}5 \pm j0{,}87)\omega_0$
4	$(-0{,}919 \pm j0{,}394)\omega_0, (-0{,}381 \pm j0{,}924)\omega_0$
5	$-\omega_0, (-0{,}812 \pm j0{,}583)\omega_0, (-0{,}308 \pm j0{,}951)\omega_0$
6	$(-0{,}959 \pm j0{,}284)\omega_0, (-0{,}714 \pm j0{,}714)\omega_0, (-0{,}258 \pm j0{,}966)\omega_0$

Graphically, this distribution of roots for a dynamic system of n-th order is shown in Fig. 2.13 – 2.18.

Normalized transition functions for a control system whose characteristic equation roots follow the Butterworth distribution are presented in Fig. 2.19.

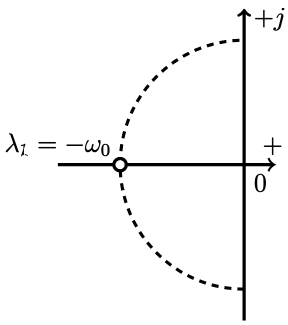

Fig. 2.13. Root Distribution of the Butterworth Polynomial for n=1

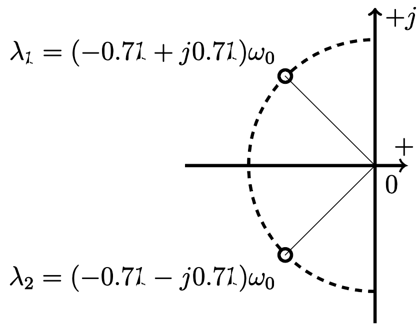

Fig. 2.14. Root Distribution of the Butterworth Polynomial for *n* = 2

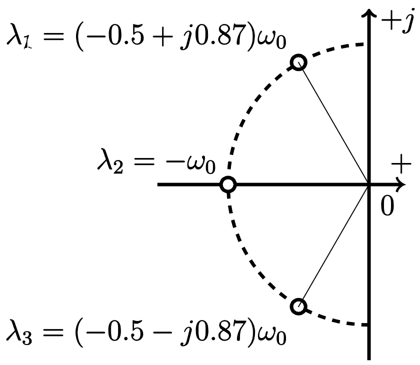

Fig. 2.15. Root Distribution of the Butterworth Polynomial for *n* = 3

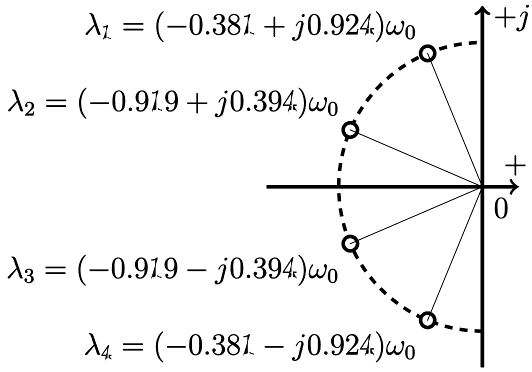

Fig. 2.16. Root Distribution of the Butterworth Polynomial for *n* = 4

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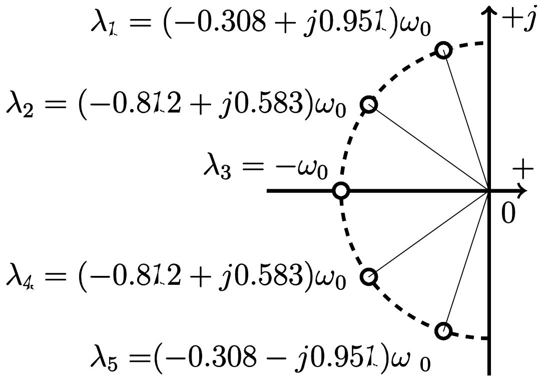

Fig. 2.17. Root Distribution of the Butterworth Polynomial for $n = 5$

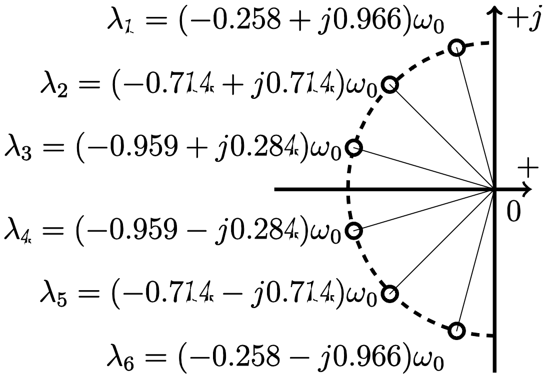

Fig. 2.18. Root Distribution of the Butterworth Polynomial for $n = 6$

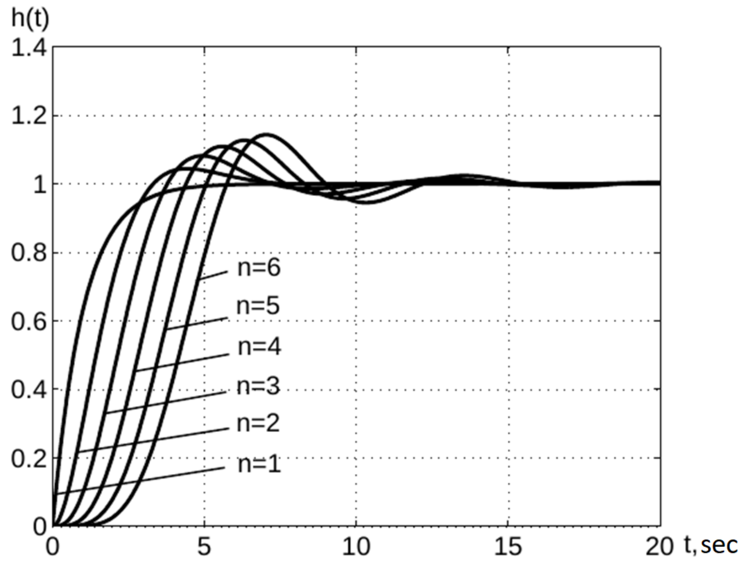

Fig. 2.19. Normalized transition functions of a control system with the Butterworth polynomial

As can be seen from this figure, the overshoot of the normalized standard transient functions is less than 20%.

The Butterworth and binomial distributions are symmetric coefficient distributions.

As previously mentioned, the concept of an optimal transition process is closely tied to minimizing of the integral functional, which characterizes the quality of the transition process. One of the simplest functionals characterizing transient processes in a control system is the integral of the squared control error

$$I_1 = \int_0^\infty (\Delta y(t))^2 \, dt. \qquad (2.429)$$

The functional (2.156) prevents the prolonged existence of control errors, and the square in the integrand is used to eliminate the influence of the error's sign on its integral value. It is evident that the dynamics and statics of the control system under constant external influences are determined by its parameters. Therefore, the functional (2.156) can be presented as an explicit function of the parameters of the control object

$$I = \frac{1}{2\omega_0} F(a_1, a_2, \cdots, a_{n-1}), \qquad (2.430)$$

where

$$F(.) = \frac{\begin{vmatrix} a_{n-1} & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\ -1 & A_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ 0 & 1 & a_{n-2} & a_{n-4} & \cdots \\ 0 & 0 & a_{n-1} & a_{n-3} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}}{\begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\ -1 & A_{n-2} & a_{n-4} & a_{n-6} & \cdots \\ 0 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ 0 & -1 & a_{n-2} & a_{n-4} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}} \qquad (2.431)$$

By minimizing this functional over all parameters a_i, one can find the standard forms of the left-hand side of the normalized characteristic equation. These standard forms for systems from the first to the sixth order are given in Table 2.5.

The roots of the characteristic polynomials indicated in Table 2.5, are provided in Table 2.6.

On the complex plane, these roots are shown in Fig. 2.20 – 2.25. Analysis of the given expressions and dependencies shows that the parameter ω_0 still does not affect the relative damping coefficient ε, but determines the duration of the transient process.

Table 2.5. Polynomials that minimize the integral of the squared control error

n	Polynomial
1	$p+\omega_0$
2	$p^2+\omega_0 p+\omega_0^2$
3	$p^3+2\omega_0^2 p+\omega_0 p^2+\omega_0^3$
4	$p^4+2\omega_0^3 p+3\omega_0^2 p^2+\omega_0 p^3+\omega_0^4$
5	$p^5+3\omega_0^4 p+3\omega_0^3 p^2+4\omega_0^2 p^3+\omega_0 p^4+\omega_0^5$
6	$p^6+3\omega_0^5 p+6\omega_0^4 p^2+4\omega_0^3 p^3+5\omega_0^2 p^4+\omega_0 p^5+\omega_0^6$

Table 2.6. Roots of polynomials that minimize the integral of the squared control error

n	Roots of a polynomial
1	$-\omega_0$
2	$(-0{,}5\pm j0{,}87)\omega_0$
3	$-0{,}57\omega_0, (-0{,}215\pm j1{,}31)\omega_0$
4	$(-0{,}395\pm j0{,}505)\omega_0, (-0{,}105\pm j1{,}57)\omega_0$
5	$-0{,}41\omega_0, (-0{,}235\pm j0{,}88)\omega_0, (-0{,}06\pm j1{,}7)\omega_0$
6	$(-0{,}315\pm j0{,}362)\omega_0, (-0{,}155\pm j1{,}5)\omega_0, (-0{,}03\pm j1{,}78)\omega_0$

The choice of this parameter is determined by the required system responsiveness and the capability to ensure a sufficient range of its linearity – the larger ω_0, the higher the gain of the direct control loop and the smaller the maximum deviation at which system saturation occurs.

Since in the case under consideration the damping coefficient is less than in the case of the Butterworth distribution, the system that minimizes the functional (2.156) has a lower stability margin and is more oscillatory compared to the previously considered systems. In addition to the above distributions, standard forms are known that are obtained by minimizing the functional

$$I_2=\int_0^\infty t\,|\Delta y(t)|\,dt, \tag{2.432}$$

which is the integral of product of the absolute value of the control error $\Delta y(t)$ and time t.

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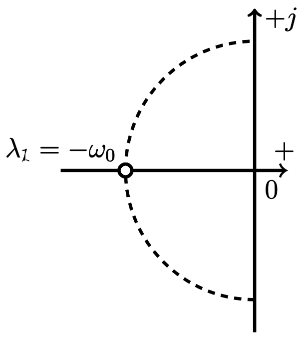

Fig. 2.20. Poles of the system, optimized according to criterion I_1 at $n = 1$

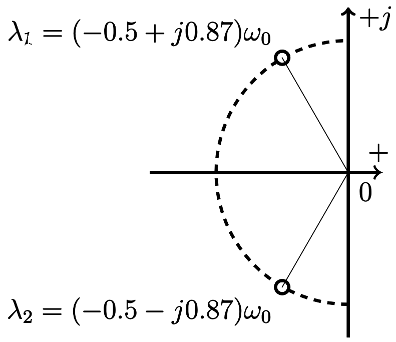

Fig.2.21. Poles of the system, optimized according to criterion I_1 at n =2

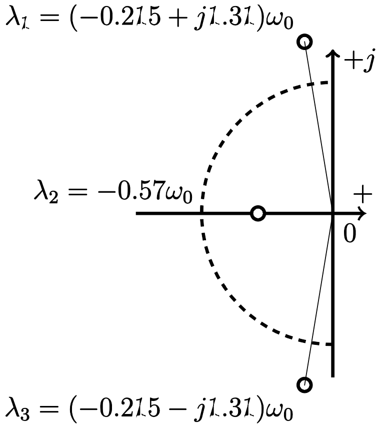

Fig. 2.22. Poles of the system, optimized according to criterion I_1 at $n = 3$

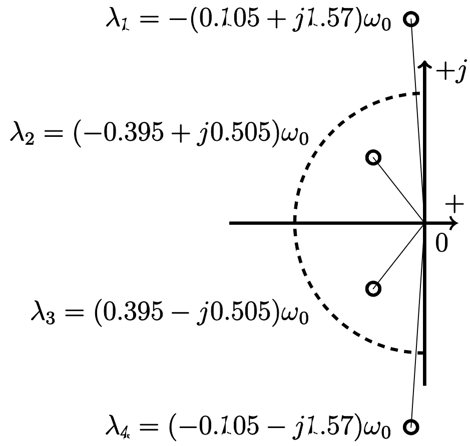

Fig. 2.23. Poles of the system, optimized according to criterion I_1 at $n = 4$

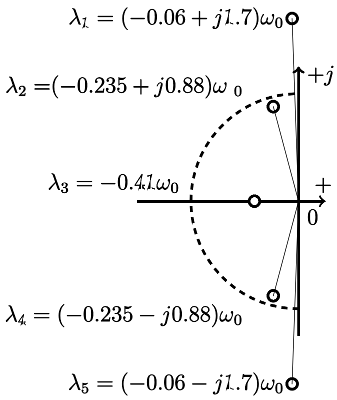

Fig. 2.24. Poles of the system, optimized according to criterion I_1 at $n = 5$

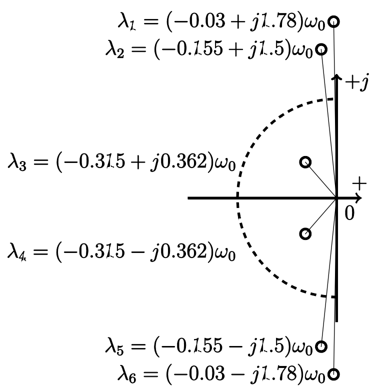

Fig. 2.25. Poles of the system, optimized according to criterion I_1 at $n = 6$

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Table 2.7. Polynomials that minimize the integral I_2

n	Polynomial
1	$p+\omega_0$
2	$p^2+1,4\omega_0 p+\omega_0^2$
3	$p^3+2,15\omega_0^2 p+1,75\omega_0 p^2+\omega_0^3$
4	$p^4+2,7\omega_0^3 p+3,4\omega_0^2 p^2+2,1\omega_0 p^3+\omega_0^4$
5	$p^5+3,4\omega_0^4 p+5,5\omega_0^3 p^2+5\omega_0^2 p^3+2,8\omega_0 p^4+\omega_0^5$
6	$p^6+3,95\omega_0^5 p+7,45\omega_0^4 p^2+8,6\omega_0^3 p^3+6,6\omega_0^2 p^4+3,25\omega_0 p^5+\omega_0^6$

Table 2.8. Roots of polynomials that minimize functional I_2

n	Roots of a polynomial
1	$-\omega_0$
2	$(-0,714\pm j0,714)\omega_0$
3	$-0,708\omega_0, (-0,521\pm j1,068)\omega_0$
4	$(-0,626\pm j0,414)\omega_0, (-0,424\pm j1,263)\omega_0$
5	$-0,896\omega_0, (-0,576\pm j0,534)\omega_0, (-0,376\pm j1,291)\omega_0$
6	$(-0,735\pm j0,287)\omega_0, (-0,581\pm j0,783)\omega_0, (-0,309\pm j1,263)\omega_0$

The characteristic polynomials and their corresponding roots for systems from the first to the sixth order are given in Tables 2.7 and 2.8, respectively. Figures 2.26 – 2.31 show a graphical interpretation of the root locations on the complex plane.

Standard forms according to Table 2.7 are widely used in practice, although their broader application is hampered by the lack of a clear algorithm for compiling these forms – they were obtained empirically [34].

It should be noted that the considered characteristic polynomials are not universal, since they provide the desired occurrence of transient processes only for control objects whose transfer function numerator is constant value. However, even with other types of numerators, these forms are very useful as a starting point for finding the optimal root locations.

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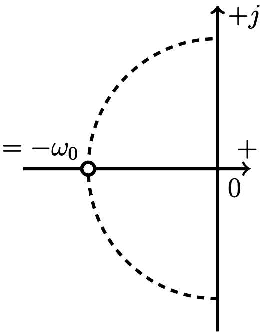

Fig. 2.26. Poles of the system, optimized according to criterion I_2 at $n = 1$

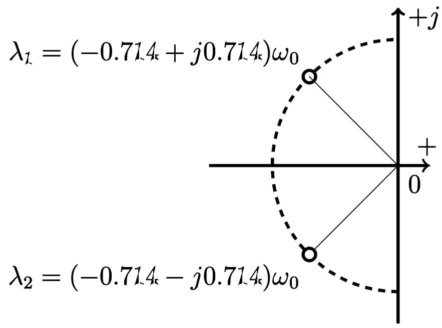

Fig. 2.27. Poles of the system, optimized according to criterion I_2 at $n = 2$

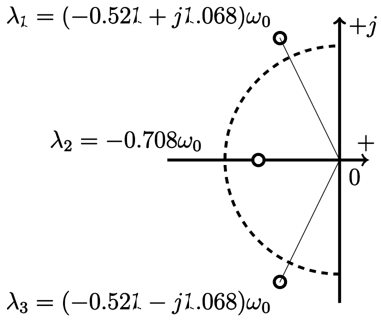

Fig. 2.28. Poles of the system, optimized according to criterion I_2 at $n = 3$

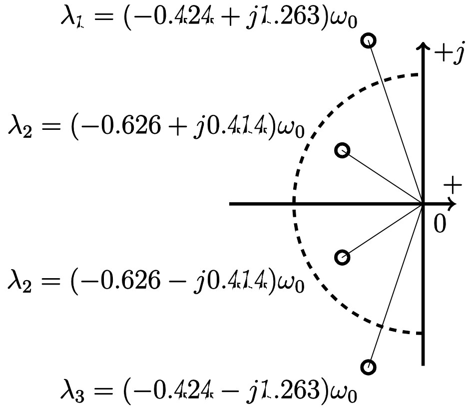

Fig. 2.29. Poles of the system, optimized according to criterion I_2 at $n = 4$

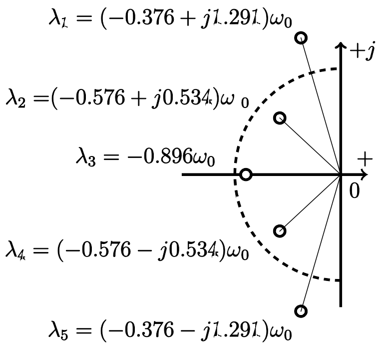

Fig. 2.30. Poles of the system, optimized according to criterion I_2 at $n = 5$

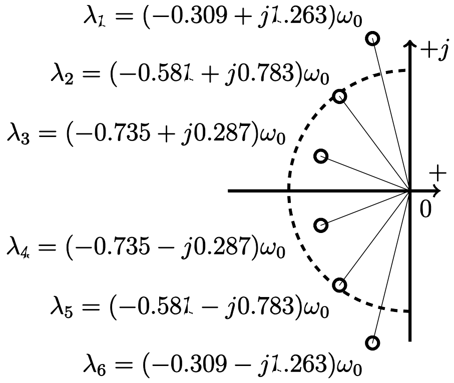

Fig. 2.31. Poles of the system, optimized according to criterion I_2 at $n = 6$

The procedure for finding the necessary characteristic polynomials consists of the following steps:

• Determination of the order of the desired characteristic polynomial, which should match the order of the control object.

• Selection of the type of desired polynomial, based on the specified overshoot and the order of the control object.

• Construction of a normalized transition function

$$W(p) = \left(D^*(p)\right)^{-1}, \quad (2.433)$$

where $D^*(p)$ is the desired characteristic polynomial corresponding to the chosen order.

• Determination of the transition process time t^*.

• Finding the parameter ω_0 based on the desired transient process time

$$\omega_0 = t^* / t \quad (2.434)$$

• Finding the coefficients of the necessary characteristic polynomials.

As a result of performing the steps outlined above, the necessary characteristic polynomial is found.

2.3.3. Algorithm for synthesizing a modal controller in canonical coordinate space

Modal control synthesis in the canonical phase space is simple and is carried out according to the following algorithm:

• The dynamics of the control object is expressed in the form of linear or linearized differential equations in normal form

$$py_j = \sum_{i=1}^{n} a_{ij} y_i + b_n u, \quad (2.435)$$

Here and below, a system with one input and one output is considered.

• The dynamics equation of object (2.162) is transformed into canonical form using a non-singular transformation

$$p\hat{y}_j = \hat{y}_{j+1};\ j \in (1, n-1);$$
$$p\hat{y}_n = -\sum_{i=1}^{n} \tilde{a}_i \hat{y}_i + \tilde{b}_n \hat{u}, \tag{2.436}$$

where the coefficients $\hat{a}_i$ are the coefficients of the characteristic polynomial of the object (2.162).

• A matrix of object coefficients is compiled

$$\mathbf{A} = \begin{pmatrix} \tilde{a}_1 & \tilde{a}_2 & \cdots & \tilde{a}_n \end{pmatrix}. \tag{2.437}$$

• The desired polynomial is selected

$$p^n + c_1 p^{n-1} + \ldots + c_n = 0. \tag{2.438}$$

• The coefficient matrix of the desired polynomial is compiled

$$\mathbf{C} = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}. \tag{2.439}$$

• The matrix of controller coefficients is determined

$$\begin{aligned} \mathbf{K} &= \begin{pmatrix} \tilde{k}_1 & \tilde{k}_2 & \cdots & \tilde{k}_n \end{pmatrix} = A - C = \\ &= \begin{pmatrix} \tilde{a}_1 & \tilde{a}_2 & \cdots & \tilde{a}_n \end{pmatrix} - \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix} = \\ &= \begin{pmatrix} \tilde{a}_1 - c_1 & \tilde{a}_2 - c_2 & \cdots & \tilde{a}_n - c_n \end{pmatrix}. \end{aligned} \tag{2.440}$$

• The controller coefficients are determined

$$\tilde{k}_1 = \tilde{a}_1 - c_1;\ \tilde{k}_2 = \tilde{a}_1 - c_2;\ \cdots;\tilde{k}_n = \tilde{a}_n - c_n. \tag{2.441}$$

• The desired control law is recorded

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$$\hat{u} = \mathbf{K}(y - y^*) =$$

$$= \begin{pmatrix} \tilde{k}_1 & \tilde{k}_2 & \cdots & \tilde{k}_n \end{pmatrix} \left[\begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \cdots \\ \hat{y}_n \end{pmatrix} - \begin{pmatrix} \hat{y}_1^* \\ \hat{y}_2^* \\ \cdots \\ \hat{y}_n^* \end{pmatrix} \right] =$$

$$= \begin{pmatrix} \tilde{k}_1 & \tilde{k}_2 & \cdots & \tilde{k}_n \end{pmatrix} \begin{pmatrix} \hat{y}_1 - \hat{y}_1^* \\ \hat{y}_2 - \hat{y}_2^* \\ \cdots \\ \hat{y}_n - \hat{y}_n^* \end{pmatrix} =$$

$$= \tilde{k}_1(\hat{y}_1 - \hat{y}_1^*) + \tilde{k}_2(\hat{y}_2 - \hat{y}_2^*) + \cdots + \tilde{k}_n(\hat{y}_n - \hat{y}_n^*), \quad (2.442)$$

where $\hat{y}_i^*$ are the desired values of the i-th state variable. Under constant input y_1^*, the remaining control inputs take zero values due to the system (2.163), so the control input (2.169) can be written as follows

$$\hat{u} = \tilde{k}_1(\hat{y}_1 - \hat{y}_1^*) + \tilde{k}_2\hat{y}_2 + \cdots + \tilde{k}_n\hat{y}_n. \quad (2.443)$$

2.3.4. Algorithm for synthesizing a modal controller in normal coordinate space

The beginning of the control synthesis algorithm in phase spaces other than canonical one is the same as the canonical, i.e. the first six steps are fully repeated:

• The dynamics of the control object is expressed in the form of linear or linearized differential equations in normal form.

• The dynamics equation of control object is transformed into canonical form using a non-singular transformation.

• A matrix of object coefficients is compiled.

• The desired polynomial is selected.

• The coefficient matrix of the desired polynomial is compiled.

• The matrix of controller coefficients is determined.

To obtain the desired control law after completing the above steps, the following additional steps are taken:

• Matrices of object coefficients under control inputs are compiled:

- for an object in a normal basis

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & \cdots & b_n \end{pmatrix}^T; \tag{2.444}$$

- for an object in the canonical basis

$$\tilde{\mathbf{B}} = \begin{pmatrix} 0 & 0 & \cdots & \tilde{b}_n \end{pmatrix}^T. \tag{2.445}$$

• A matrix of object controllability is compiled:

- based on its equations in the normal basis

$$\mathbf{Y} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}. \tag{2.446}$$

- based on its equations in the canonical basis

$$\tilde{\mathbf{Y}} = \begin{bmatrix} \tilde{\mathbf{B}} & \tilde{\mathbf{A}}\tilde{\mathbf{B}} & \tilde{\mathbf{A}}^2\mathbf{B} & \dots & \tilde{\mathbf{A}}^{n-1}\mathbf{B} \end{bmatrix}. \tag{2.447}$$

• The transformation matrix is calculated

$$\mathbf{P} = \mathbf{Y}\mathbf{Y}^{-1}. \tag{2.448}$$

• The controller coefficients are calculated

$$\mathbf{K}' = \left(\mathbf{K}^T\mathbf{P}\right)^T. \tag{2.449}$$

• The desired control law is formulated

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$$u = \mathbf{K'(y - y^*)} =$$

$$= \begin{pmatrix} k_1 & k_2 & \cdots & k_n \end{pmatrix} \left[\begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix} - \begin{pmatrix} y_1^* \\ y_2^* \\ \cdots \\ y_n^* \end{pmatrix} \right] =$$

$$= \begin{pmatrix} k_1 & k_2 & \cdots & k_n \end{pmatrix} \begin{pmatrix} y_1 - y_1^* \\ y_2 - y_2^* \\ \cdots \\ y_n - y_n^* \end{pmatrix} =$$

$$= K_1(y_1 - y_1^*) + k_2(y_2 - y_2^*) + \cdots + k_n(y_n - y_n^*). \quad (2.450)$$

The implementation of these steps is not limited only to the case of controller synthesis in a normal basis, but extends to any phase spaces.

2.4. Modal Control Communication with modified symmetry principle

Analysis of control inputs synthesized using model control methods shows that they match, with accuracy up to weighting coefficients, the control algorithms as a result of solving the problem of analytical design of controllers. This creates the basis for establishing a relationship between these methods. To do this, let us consider the equations of perturbed motion of a generalized third-order dynamic object, presented in canonical form

$$p\eta_1 = \eta_2; \quad p\eta_2 = \eta_3;$$
$$p\eta_3 = -a_1\eta_1 - a_2\eta_2 - a_3\eta_3 + M_3 U. \quad (2.451)$$

We will assume that the object (2.178) is subject to optimal control

$$U = -g(v_{13}\eta_1 + v_{23}\eta_2 + v_{33}\eta_3), \quad (2.452)$$

where g is the gain of the controller, v_{i3} are the coefficients of the Lyapunov function

$$V = v_{11}\eta_1^2 + 2v_{12}\eta_1\eta_2 + 2v_{13}\eta_1\eta_3 + \\ + v_{22}\eta_2^2 + 2v_{23}\eta_2\eta_3 + v_{33}\eta_3^2, \qquad (2.453)$$

that minimize the integral quality functional

$$I = \int_0^\infty \left[(v_{13}\eta_1 + v_{23}\eta_2 + v_{33}\eta_3)^2 + cU^2 \right] dt. \qquad (2.454)$$

Minimizing the functional (2.181) on the trajectories of the perturbed motion (2.178) guarantees the asymptotic tendency of all coordinates of the perturbed motion to zero.

Substituting the control input (2.179) into the system (2.178), yields the equations of perturbed motion of the closed-loop control system

$$p\eta_1 = \eta_2; \\ p\eta_2 = \eta_3; \\ p\eta_3 = -a_1\eta_1 - a_2\eta_2 - a_3\eta_3 - M_3 g(V_{13}\eta_1 + V_{23}\eta_2 + V_{33}\eta_3) \qquad (2.455)$$

or after collecting terms in the last equation

$$p\eta_1 = \eta_2; \\ p\eta_2 = \eta_3; \\ p\eta_3 = -(a_1 + M_3 gV_{13})\eta_1 - (a_2 + M_3 gV_{23})\eta_2 - \\ - (a_3 + M_3 gV_{33})\eta_3. \qquad (2.456)$$

Substituting the first and second equations into the third equation of the system (2.183), we obtain a third-order equation describing the dynamics of the closed-loop system

$$(a_1 + M_3 g v_{13})\eta_1 + (a_2 + M_3 g v_{23}) p\eta_1 + \\ + (a_3 + M_3 g v_{33}) p^2 \eta_1 + p^3 \eta_1 = 0. \qquad (2.457)$$

Dividing both sides of equation (2.184) by η_1, we obtain the characteristic equation of the closed-loop system

$$(a_1 + M_3 g v_{13}) + (a_2 + M_3 g v_{23}) p + \\ + (a_3 + M_3 g v_{33}) p^2 + p^3 = 0. \qquad (2.458)$$

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The coefficients v_{i3} of the Lyapunov function (2.180) are determined through the parameters of the control object (2.178) [15] by the following dependencies

$$v_{33} = 1; v_{23} = a_3; v_{13} = a_2, \tag{2.459}$$

therefore, the final form of the characteristic equation is

$$(a_1 + M_3 g a_2) + (a_2 + M_3 g a_3) p + \\ +(a_3 + M_3 g) p^2 + p^3 = 0. \tag{2.460}$$

Polynomials (2.185) and (2.187) illustrate a direct transition from the analytical design of controllers to modal control. However, it is also possible to show the reverse transition by selecting a certain polynomial

$$b_0 + b_1 p + b_2 p^2 + p^3 = 0, \tag{2.461}$$

the distribution of roots p_j of which ensures the desired quality of the transition process. Equating the coefficients for the same degrees p of polynomials (2.185) and (2.188)

$$a_1 + M_3 g V_{13} = b_1; \\ a_2 + M_3 g V_{23} = b_2; \\ a_3 + M_3 g V_{33} = b_3, \tag{2.462}$$

it is possible to determine the weighting coefficients of the quality functional (2.181) and control input (2.179)

$$v_{13} = \frac{b_1 - a_1}{M_3 g}; v_{23} = \frac{b_2 - a_2}{M_3 g}; v_{33} = \frac{b_3 - a_3}{M_3 g}. \tag{2.463}$$

The polynomial (2.187) can be generalized in case of a dynamic object of arbitrary order

$$\sum_{i=1}^{n-1} (a_i + M_n g a_{i+1}) p^{i-1} + (a_n + M_n g) p^{n-1} + p^n = 0. \tag{2.464}$$

Using a polynomial (2.191) as a desired one in modal synthesis guarantees the asymptotic stability of the perturbed motion.

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Generalization polynomials (2.191) and (2.185) for the case of control of an object of arbitrary order with arbitrary coefficients of the Lyapunov function leads to the following characteristic polynomial

$$\sum_{i=1}^{n}(a_i + M_n g v_{in}) p^{i-1} + p^n = 0, \tag{2.465}$$

which, depending on the set control goal, ensures that the closed-loop system has the desired characteristics. At the same time, the dependence of the coefficients of the found polynomials on the gain factor g is of particular interest. This factor creates the prerequisites for the synthesis of discontinuous systems without the use of complex mathematical methods.

2.5. Regularization of dynamical systems

Unlike modal control, the use of which to synthesize control inputs depending on the selected desired polynomial may lead to overshoots in a closed-loop EMS, the use of a modified symmetry principle eliminates the occurrence of oscillatory processes in closed-loop systems. However, the drawback of solving the ADC problem using the modified symmetry principle is the inability to synthesize control inputs for dynamic objects with more than one zero root of the characteristic equation without changing the structure of the control object.

From the perspective of ADC, such a problem is incorrectly posed and to solve it, regularization must be used, that is, add some additional information to the mathematical description of the electromechanical object [24]. For an EMS of the n-th order, whose dynamics is represented by equations in Brunovsky controllable form, such additional information can be a control algorithm for a closed-loop system of $n-1$ order. This algorithm is called regularizing.

At the same time, if regularizing control actions are synthesized in each loop by solving the problem of ADC, there are two possible methods for determining the desired control input. These options lead to sequential and parallel regularization, respectively.

In sequential regularization, the control input applied to the object is formed by sequentially switched on state variable controls from 2 to n (Fig. 2.32). The input of the internal i controller is the output $i-1$, i.e. the principle of subordinate control is implemented.

In parallel regularization, all regularizing controllers are switched on in parallel, and the reference signals for each controller are determined by the original input signal and its derivatives (Fig. 2.33), or if we assume that this signal is constant, then the reference inputs on the internal regularization loops should be taken as zero (Fig. 2.34).

Therefore, the regularization of electromechanical objects in the Brunovsky form is reduced to a sequential synthesis of the control inputs of $n-1$ state variable with subsequent substitution of the obtained control algorithms into the equations of the object's dynamics.

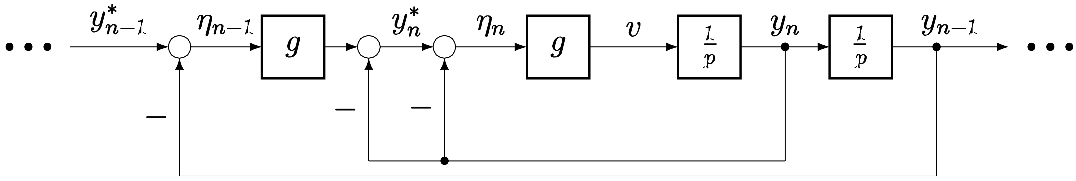

Fig. 2.32. Sequential regularization principle

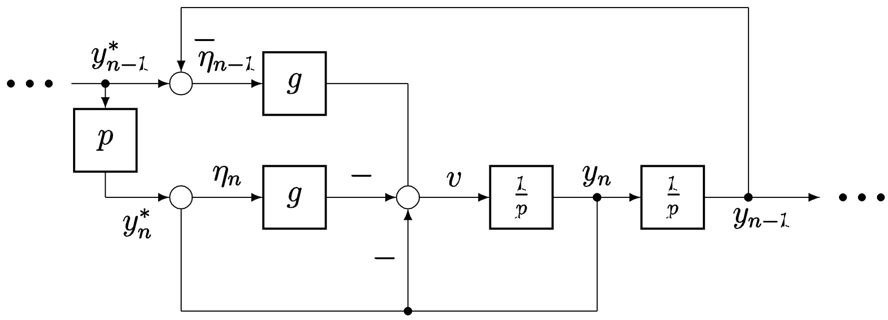

Fig. 2.33. Parallel regularization principle

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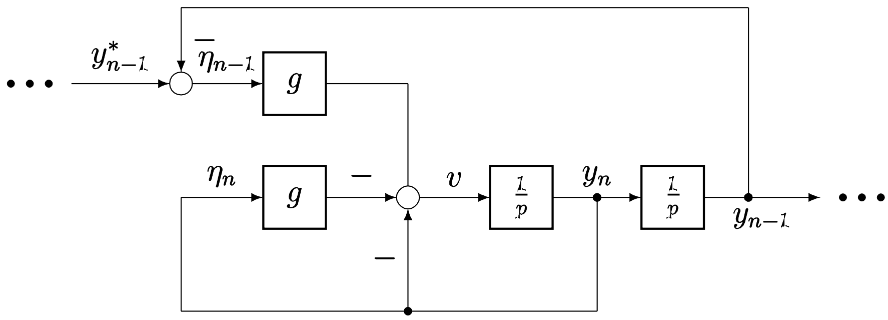

Rice. 2.34. The principle of parallel regularization with a constant reference input

Let us consider the regularization of the motion equations of a third-order dynamic object in the Brunovsky controllable form

$$py_1 = y_2;$$
$$py_2 = y_3;$$
$$py_3 = \nu_3. \qquad (2.466)$$

The characteristic equation of the considered object has three zero roots and therefore, from the point of view of solving the ADC problem, is incorrect, since the latter requires the characteristic polynomial of the control object to have at most one zero root. Let us resolve this contradiction by synthesizing control inputs using the internal coordinates of the object. Substituting the obtained control algorithms into the original system of equations allows us to eliminate the incorrectness of the object description. We will call such a procedure *regularization*.

Let us isolate the last equation from the equations (2.193)

$$py_3 = \nu_3 \qquad (2.467)$$

and find a control input that guarantees asymptotic stability of the closed-loop system. Such a control input is control of the form

$$\nu_3 = g\left(y_3^* - y_3\right). \qquad (2.468)$$

Substituting the control algorithm (2.195) into the motion equations (2.193), we obtain the last two equations

$$py_1 = y_2;\ py_2 = y_3;\ py_3 = g\left(y_3^* - y_3\right). \qquad (2.469)$$

Now, let us proceed to synthesizing of variable control y_2. Unlike the previous case, the determination of the control input by coordinate y_2 is ambiguous and is associated with the possibility of connecting controllers in sequentially and in parallel. In the case of a sequential connection of controls, the control input $\overset{*}{y}_3$ is formed by a variable regulator y_2, which implements the following algorithm

$$\overset{*}{y}_3 = g\left(g\left(\overset{*}{y}_2 - y_2\right) - y_3\right) = g^2\overset{*}{y}_2 - g^2 y_2 - g y_3. \qquad (2.470)$$

When connecting controls in parallel, the system (2.196) must be supplemented with an input ν_2

$$p y_1 = y_2;\; p y_2 = y_3;\; p y_3 = g\overset{*}{y}_3 - g y_3 + \nu_2, \qquad (2.471)$$

which is determined by the expression

$$\nu_2 = g(g(\overset{*}{y}_2 - y_2) - y_3) + g\overset{*}{y}_3. \qquad (2.472)$$

Let us determine the control inputs applied to the object. For sequential inclusion, we substitute the expression (2.197) into the equation (2.195)

$$\nu_3 = g(g(g(\overset{*}{y}_2 - y_2) - y_3) + g\overset{*}{y}_3 - y_3), \qquad (2.473)$$

and for parallel – we sum up the expressions (2.200) and (2.195)

$$\nu_2 + \nu_3 = g^2\overset{*}{y}_2 - g^2 y_2 - 2g y_3. \qquad (2.474)$$

Expressions (2.200) and (2.201) form various forms of the last equation of the system (2.193) and make it possible to use the resulting equations to synthesize a coordinate controller y_1 by solving the ADC problem. Depending on the selected scheme for connecting internal controls, the parameters of the controller coordinate y_1 and its connection scheme with internal controllers will be different.

CHAPTER 3
SLIDING MODES IN ELECTROMECHANICAL SYSTEMS

3.1. Concept of sliding mode

The operating mode of relay systems is a self-oscillatory mode, and under certain conditions the systems begin to move along certain degenerate trajectories, on which their properties change significantly – *a sliding mode* appears.

The *ideal sliding mode* is understood as a movement where the representing point oscillates relative to the sliding surface with an infinitely high frequency and an infinitely small amplitude. The actual sliding movement, due to various “imperfections” of the switching device, occurs with a finite frequency and finite amplitude. For a system where sliding mode occurs to be stable, three conditions must be met:

- sliding condition;
- condition for hitting the switching line;
- condition for stability of movement along the switching line.

Let us consider these conditions in detail for a generalized dynamic object, the dynamics of which is described by nonlinear equations in matrix form:

$$p\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}, U), \tag{3.475}$$

while the object is subjected to discontinuous control

$$u = \begin{cases} u^{+}(S(\boldsymbol{\eta})), & if \quad S(\boldsymbol{\eta}) > 0; \\ u^{-}(S(\boldsymbol{\eta})), & if \quad S(\boldsymbol{\eta}) < 0, \end{cases} \tag{3.476}$$

where $\mathbf{f}(\boldsymbol{\eta}, U)$ is a discontinuous function, $u^{+}(S(\boldsymbol{\eta})), u^{-}(S(\boldsymbol{\eta}))$ are continuous functions, with $u^{+}(S(\boldsymbol{\eta})) \neq u^{-}(S(\boldsymbol{\eta}))$, $S(\boldsymbol{\eta})$ representing the equation of the switching line.

3.1.1. Conditions for the emergence of sliding mode and hitting the switching line

In the sliding mode, the switching line obeys the relation

$$S(\boldsymbol{\eta}) = 0 \tag{3.477}$$

and divides the state space R^n into two subspaces: subspace $R_-^n = \{\boldsymbol{\eta} : S(\boldsymbol{\eta}) < 0\}$ and subspace $R_+^n = \{\boldsymbol{\eta} : S(\boldsymbol{\eta}) > 0\}$. In each of these subspaces, continuous functions $\mathbf{f}_-(\boldsymbol{\eta}, U)$ and $\mathbf{f}_+(\boldsymbol{\eta}, U)$ are defined. For these functions the expressions for the left and right limits are valid

$$\mathbf{f}_-(\boldsymbol{\eta}, U) = \lim_{S(\boldsymbol{\eta}) \to -0} \mathbf{f}(\boldsymbol{\eta}, U);$$
$$\mathbf{f}_+(\boldsymbol{\eta}, U) = \lim_{S(\boldsymbol{\eta}) \to +0} \mathbf{f}(\boldsymbol{\eta}, U); \tag{3.478}$$

Functions (3.4) determine the movement speed through the switching line

$$\frac{dS(\boldsymbol{\eta})}{dt} = \frac{dS(\boldsymbol{\eta})}{d\boldsymbol{\eta}} \mathbf{f}(\boldsymbol{\eta}, U) = \nabla S(\boldsymbol{\eta}) \mathbf{f}(\boldsymbol{\eta}, U). \tag{3.479}$$

The gradient of the function $S(\boldsymbol{\eta})$ is always directed normal to the surface S in the direction of increasing the function $S(\boldsymbol{\eta})$ (Fig. 3.1).

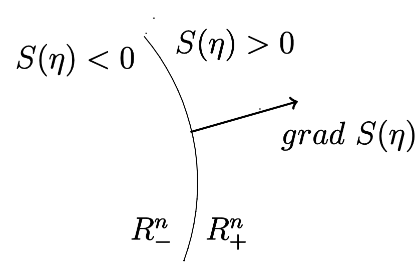

Fig. 3.1. Switching function gradient

Therefore, to evaluate the occurrence of sliding mode, one can use the limit

$$pS^{-}(\boldsymbol{\eta}) = \lim_{S(\boldsymbol{\eta})\to -0} pS(\boldsymbol{\eta}) = \nabla S(\boldsymbol{\eta}) \times \mathbf{f}^{-}(\boldsymbol{\eta}, U),$$
$$\mathbf{f}^{-}(\boldsymbol{\eta}, U) = \mathbf{f}(\boldsymbol{\eta}, U^{-}). \qquad (3.480)$$

Let us consider the limit (3.6). If the speed of movement $pS^{-}(\boldsymbol{\eta})$ is positive, then the angle between vectors $\nabla S(\boldsymbol{\eta})$ and $\mathbf{f}^{-}(\boldsymbol{\eta}, U)$ is acute, which means the vector $\mathbf{f}^{-}(\boldsymbol{\eta}, U)$ is directed towards the subspace R_{+}^{n} (Fig. 3.2) and vice versa, if the speed $pS^{-}(\boldsymbol{\eta})$ is negative, then the vector $\mathbf{f}^{-}(\boldsymbol{\eta}, U)$ is directed towards the subspace R_{-}^{n} (Fig. 3.3).

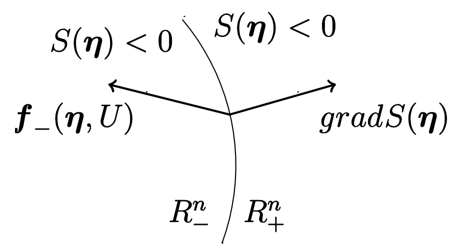

Fig. 3.2. Switching function when $pS^{-}(\boldsymbol{\eta}) < 0$

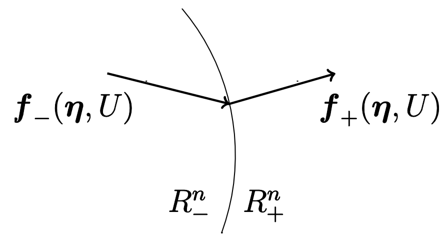

Fig. 3.3. Switching function when $pS^{-}(\boldsymbol{\eta}) > 0$

Similar statements are true for the limit

$$pS^{+}(\boldsymbol{\eta}) = \lim_{S(\boldsymbol{\eta})\to +0} pS(\boldsymbol{\eta}) = \nabla S(\boldsymbol{\eta}) \times \mathbf{f}^{+}(\boldsymbol{\eta}, U),$$
$$\mathbf{f}^{+}(\boldsymbol{\eta}, U) = \mathbf{f}(\boldsymbol{\eta}, U^{+}). \qquad (3.481)$$

Depending on the signs of the product $pS^{+}(\mathbf{\eta})\times pS^{-}(\mathbf{\eta})$, there are three types of points on the switching line:

• Points of the phase trajectory where the product $pS^{+}(\mathbf{\eta})\times pS^{-}(\mathbf{\eta})$ is positive, i.e.

$$pS^{+}(\mathbf{\eta})\times pS^{-}(\mathbf{\eta})>0. \qquad (3.482)$$

At these points, the vectors $\mathbf{f}^{-}(\mathbf{\eta},U)$ and $\mathbf{f}^{+}(\mathbf{\eta},U)$ are directed in same direction (Fig. 3.4 – 3.5).

This type of motion occurs when controlling objects whose dynamics are described by differential equations with even functions on the right-hand sides, i.e.

$$\mathbf{f}\ (\mathbf{\eta},U)=\mathbf{f}\ (-\mathbf{\eta},U)=\mathbf{f}\ (\mathbf{\eta},-U)=\mathbf{f}\ (-\mathbf{\eta},-U). \qquad (3.483)$$

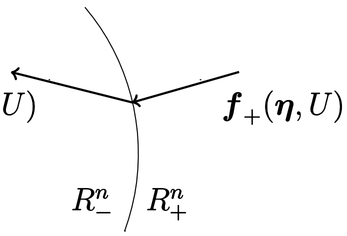

Fig. 3.4. Points on the switching surface

$$pS^{+}(\mathbf{\eta})\times pS^{-}(\mathbf{\eta})>0$$

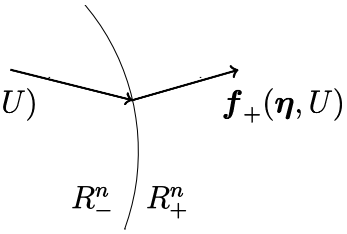

Fig. 3.5. Points on the switching surface

$$pS^{+}(\mathbf{\eta})\times pS^{-}(\mathbf{\eta})>0$$

The switching line does not have a significant effect on the nature of the object's movement, because the components of the vector of rates of change of state variables $p\boldsymbol{\eta}$ do not change sign.

• Points where $pS^+(\boldsymbol{\eta}) > 0$ and $pS^-(\boldsymbol{\eta}) < 0$.

At these points, the vectors $f^+(\boldsymbol{\eta}, U)$ and $f^-(\boldsymbol{\eta}, U)$ are directed in different directions: $\mathbf{f}^-(\boldsymbol{\eta}, U)$ is directed towards the subspace R_-^n, while $\mathbf{f}^+(\boldsymbol{\eta}, U)$ is directed towards the subspace R_+^n (Fig. 3.6).

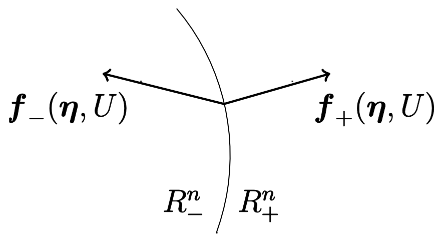

Fig. 3.6. Points on the switching surface

$pS^+(\boldsymbol{\eta}) > 0$ and $pS^-(\boldsymbol{\eta}) < 0$

In this case, if the object starts moving in the vicinity of the switching line, it will move away from the switching line, and such movement will not be accompanied by a change in the sign of the control input. An example of a control system where this type of motion may occur is a positive feedback system.

• Points where the following conditions are met:

$$\begin{aligned} pS^+(\boldsymbol{\eta}) &= \nabla S(\boldsymbol{\eta}) \times \mathbf{f}^+(\boldsymbol{\eta}, U) < 0; \\ pS^-(\boldsymbol{\eta}) &= \nabla S(\boldsymbol{\eta}) \times \mathbf{f}^-(\boldsymbol{\eta}, U) > 0. \end{aligned} \quad (3.484)$$

In this case, the vectors $f^+(\boldsymbol{\eta}, U)$ and $f^-(\boldsymbol{\eta}, U)$ are directed towards the switching surface, i.e. phase trajectories near the surface $S(\boldsymbol{\eta})$ are directed oppositely (Fig. 3.7).

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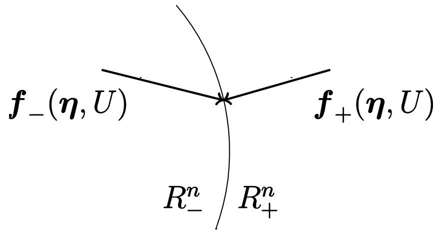

Fig. 3.7. Points on the switching surface

$$pS^+(\mathbf{\eta}) < 0 \text{ and } pS^-(\mathbf{\eta}) > 0$$

In the case where the representing point reaches such trajectories and reaches the switching line, the sliding process begins. The equations (3.10) describe the existence of the sliding mode, and the surface $S(\mathbf{\eta})$ is the sliding surface. Analysis of Fig. 3.7 shows that in order to achieve the switching line, it is necessary and sufficient that at $S(\mathbf{\eta}) > 0$ the derivative $pS(\mathbf{\eta})$ should be negative, and at $S(\mathbf{\eta}) < 0$ the derivative $pS(\mathbf{\eta})$ should be positive. The condition for the representing point to hit the sliding surface is determined by the expression

$$S(\mathbf{\eta}) \times pS(\mathbf{\eta}) < 0. \quad (3.485)$$

3.1.2. Movement along the switching line

Since the relay system is nonlinear and the control its action in the sliding mode is not defined, the problem of mathematical description of such a system while it is operating in sliding mode.

There are two possible approaches to solving the problem of describing ideal sliding motion: the axiomatic approach and the approach based on limit transitions. Recently, the axiomatic description of sliding motion proposed by A. F. Filippov has become the most widespread [35].

According to this description, the dynamics of a closed-loop system is represented by free motion equations:

$$p\mathbf{\eta} = \mathbf{f^0}(\mathbf{\eta}), \quad (3.486)$$

where

$$\mathbf{f}^{\mathbf{0}}(\boldsymbol{\eta}) = \mu\mathbf{f}^{+}(\boldsymbol{\eta}) + (1-\mu)\mathbf{f}^{-}(\boldsymbol{\eta}),\ 0 < \mu < 1. \quad (3.487)$$

When determining the parameter μ, we will assume that the speed of movement along the switching line is directed tangent to this line, and the vector $\mathbf{f}^{\mathbf{0}}(\boldsymbol{\eta})$ lies on this tangent. Since the tangent and the normal are mutually perpendicular, then the following expression is valid for the vectors $\mathbf{f}^{\mathbf{0}}(\boldsymbol{\eta})$ and $\nabla S(\boldsymbol{\eta})$

$$\nabla S(\boldsymbol{\eta})\mathbf{f}^{\mathbf{0}}(\boldsymbol{\eta}) = 0. \quad (3.488)$$

Substituting the value of the function $\mathbf{f}^{\mathbf{0}}(\boldsymbol{\eta})$ into the equation (3.14), we obtain an equation of the form

$$\nabla S(\boldsymbol{\eta})\mathbf{f}^{\mathbf{0}}(\boldsymbol{\eta}) = \nabla S(\boldsymbol{\eta})\left[\mu\mathbf{f}^{+}(\boldsymbol{\eta}) + (1-\mu)\mathbf{f}^{-}(\boldsymbol{\eta})\right] = 0. \quad (3.489)$$

The equation (3.15) allows us to uniquely determine the unknown parameter μ

$$\mu = \frac{\nabla S(\boldsymbol{\eta})\mathbf{f}^{-}(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})(\mathbf{f}^{-}(\boldsymbol{\eta}) - (\mathbf{f}^{+}(\boldsymbol{\eta}))}, \quad (3.490)$$

and its substitution into the original system (3.12) yields the equations of motion for the system in the sliding mode

$$p\boldsymbol{\eta} = \frac{\nabla S(\boldsymbol{\eta})\mathbf{f}^{-}(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})(\mathbf{f}^{-}(\boldsymbol{\eta}) - (\mathbf{f}^{+}(\boldsymbol{\eta}))}\mathbf{f}^{+}(\boldsymbol{\eta}) - \\ - \frac{\nabla S(\boldsymbol{\eta})\mathbf{f}^{-}(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})(\mathbf{f}^{-}(\boldsymbol{\eta}) - (\mathbf{f}^{+}(\boldsymbol{\eta}))}\mathbf{f}^{-}(\boldsymbol{\eta}),\ S(\boldsymbol{\eta}(\mathbf{0})) = 0. \quad (3.491)$$

Let us illustrate how to use dependencies (3.16) and (3.17) using examples from the analysis of simple relay systems, we obtain the equations of motion of a number of elementary systems in a sliding mode.

Let us consider a relay servo drive control system (Fig. 3.8).

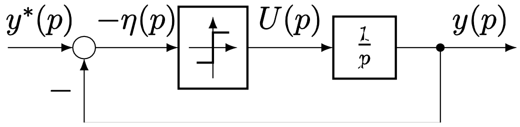

Fig. 3.8. Relay servo drive control system

The perturbed motion of this system, in the first approximation, is described by the differential equation

$$p\eta = U. \qquad (3.492)$$

Servo drive control is formed in accordance with the relay law

$$U = \text{sign}[S],\ S = -\eta. \qquad (3.493)$$

The functions $f^{-}(S(\eta))$ and $f^{+}(S(\eta))$ will be

$$f^{-}(S(\eta)) = -1,\ f^{+}(S(\eta)) = 1. \qquad (3.494)$$

Taking into account the previously found dependencies, the parameter μ is determined by the expression

$$\mu = \frac{\nabla S(\boldsymbol{\eta})\mathbf{f}^{-}(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})(\mathbf{f}^{-}(\boldsymbol{\eta}) - (\mathbf{f}^{+}(\boldsymbol{\eta}))} = \frac{-1\times(-1)}{-1\times(-1-1)} = \frac{1}{2}. \qquad (3.495)$$

Analysis of the expression (3.21) shows that in the considered control system, switching of the control input occurs with a duty cycle of 50%. The obtained result agrees well with the results of numerical analysis of relay systems.

Let us compose the equation of motion of a closed-loop system in a sliding mode

$$\begin{aligned} p\eta_1 &= \frac{\nabla S(\boldsymbol{\eta})\mathbf{f}^{-}(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})(\mathbf{f}^{-}(\boldsymbol{\eta}) - (\mathbf{f}^{+}(\boldsymbol{\eta}))}\mathbf{f}^{+}(\boldsymbol{\eta}) + \\ &+ \left(1 - \frac{\nabla S(\boldsymbol{\eta})\mathbf{f}^{-}(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})(\mathbf{f}^{-}(\boldsymbol{\eta}) - (\mathbf{f}^{+}(\boldsymbol{\eta}))}\right)\mathbf{f}^{-}(\boldsymbol{\eta}) = \\ &= \frac{1}{2}\times 1 + \left(1 - \frac{1}{2}\right)\times(-1) = 0. \end{aligned} \qquad (3.496)$$

Thus, it can be concluded that in sliding mode, the state variables of the servo

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drive control system do not change, and sliding mode occurs when zero deviation is reached.

Now let us consider a high-speed electromechanical system with an inertia-free channel for the generating of electromagnetic torque (Fig. 3.9).

The equations of perturbed motion and the control input in such a system have the form

$$p\eta = -\eta + U,$$
$$U = \text{sign}[S],\ S = -\eta, \tag{3.497}$$

and the functions $f^{-}(S(\eta))$ and $f^{+}(S(\eta))$ are governed by relations

$$f^{-}(S(\eta)) = -\eta - 1,\ f^{+}(S(\eta)) = -\eta + 1. \tag{3.498}$$

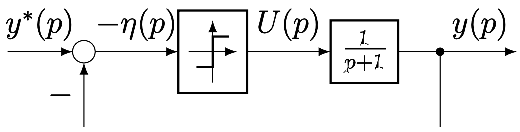

Fig. 3.9. Relay speed control system of generalized electromechanical system

Then the parameter μ is determined by the dependence

$$\mu = \frac{\nabla S(\boldsymbol{\eta})\mathbf{f}^{-}(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})(\mathbf{f}^{-}(\boldsymbol{\eta}) - (\mathbf{f}^{+}(\boldsymbol{\eta}))} = \frac{-1\times(-\eta-1)}{-1\times((-\eta-1)-(-\eta+1)} = \frac{\eta+1}{2}, \tag{3.499}$$

and the equation of motion of a closed-loop control system in a sliding mode will take the form

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$$p\eta = \frac{\nabla S(\boldsymbol{\eta})\mathbf{f}^{-}(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})(\mathbf{f}^{-}(\boldsymbol{\eta}) - (\mathbf{f}^{+}(\boldsymbol{\eta}))}\mathbf{f}^{+}(\boldsymbol{\eta}) +$$

$$+\left(1 - \frac{\nabla S(\boldsymbol{\eta})\mathbf{f}^{-}(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})(\mathbf{f}^{-}(\boldsymbol{\eta}) - (\mathbf{f}^{+}(\boldsymbol{\eta}))}\right)\mathbf{f}^{-}(\boldsymbol{\eta}) =$$

$$= \frac{\eta+1}{2}\times(-\eta+1) + \left(1 - \frac{\eta+1}{2}\right)\times(-\eta-1) =$$

$$= \frac{1-\eta^2}{2} - \frac{1-\eta^2}{2} = 0. \qquad (3.500)$$

Summarizing the examples discussed above, it can be concluded that in the sliding mode the coordinates of the perturbed motion do not change, and the sliding mode can be considered as a quasi-static operating mode of the electromechanical system.

3.1.3. Properties of systems operating in sliding mode

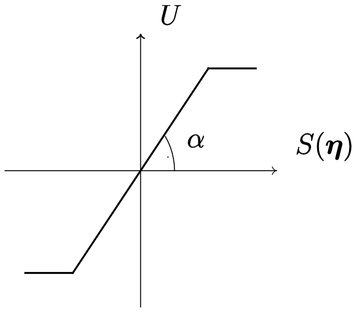
Assuming that the ideal relay element is inertia-free, we define its transfer function as follows:

$$W_{\text{sign}}(p) = \frac{U(p)}{S(\boldsymbol{\eta}(\mathbf{p}))} = \frac{\text{sign}(S(\boldsymbol{\eta}(\mathbf{p})))}{S(\boldsymbol{\eta}(\mathbf{p}))} = \frac{1}{|S(\boldsymbol{\eta}(\mathbf{p}))|}. \qquad (3.501)$$

When hitting the switching line, the gain factor of the relay controllers is determined by the limit

$$\lim_{S(\boldsymbol{\eta}(\mathbf{p}))\to 0} W_{\text{sign}}(p) = \infty \quad . \qquad (3.502)$$

Therefore, a relay controller in a sliding mode can be considered as a limiting case of a nonlinear element of the “saturation” type (Fig. 3.10).



Fin. 3.10. Nonlinear saturation type element

A nonlinear system with a limiting element (Fig. 3.10) tends to a relay system (Fig. 3.11) as the slope coefficient of the characteristic $k = \operatorname{tg} \alpha$, shown in Fig. 3.10, tends to infinity. For an arbitrarily small deviation of the signal $S(\boldsymbol{\eta})$ from zero, the control input U is constant in absolute value and changes sign with the change in the sign of the control signal, which allows us to speak about the tendency of the gain factor, defined as the ratio of the final output signal to an infinitesimal input signal, to infinity. Then, assessing the properties of a relay system in a sliding mode comes down to studying the characteristics of a linearized system obtained from a relay by replacing a relay element with a linear amplifier, the gain of which increases indefinitely (Fig. 3.12).

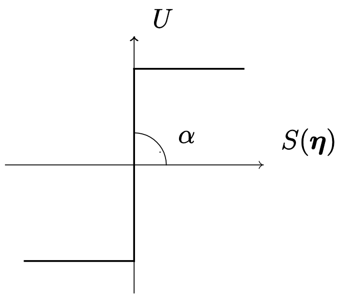


Fig. 3.11. Ideal relay element

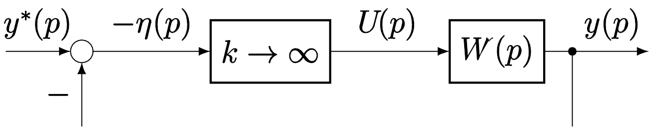


Fig. 3.12. Linearized system with infinite gain factor

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The stability of a linearized system is determined by the location of the poles of its transfer function or the roots of the corresponding characteristic equation. In this case, the linearized system will be stable if all poles of its transfer function (or all roots of the corresponding characteristic equation) have negative real parts with unlimited increase of k. Transfer function of a closed-loop system with respect to the control input $U(t)$ (Fig. 3.12) is

$$K_k(p) = \frac{k}{1 + kW(p)}. \tag{3.503}$$

Since, in the general case, the transfer function of the control object is the ratio of polynomials of the n-th and m-th degrees

$$W(p) = \frac{P(p)}{Q(p)}, \tag{3.504}$$

then

$$K_k(p) = \frac{kQ(p)}{Q(p) + kP(p)}. \tag{3.505}$$

Then the characteristic equation takes the form

$$Q(p) + kP(p) = 0, \tag{3.506}$$

or

$$\frac{1}{k}Q(p) + P(p) = 0. \tag{3.507}$$

If we tend k to infinity, then m roots of the equation (3.33) will tend to the roots of the equation $P(p) = 0$. This equation can be considered as the characteristic equation of the limit system, which is obtained from the original one in the limit at $k \to \infty$. In this case, the equation (3.33) is transformed to the form

$$P(p) = 0. \tag{3.508}$$

So, the condition for the stability of the sliding mode is the negativity of the real parts of the roots of the characteristic equation (3.34). The stability condition is one of the most important criteria for the analysis of sliding modes of relay automatic control systems.

The designing of systems that are stable with an unlimited increase in the gain due to the organization of a sliding mode is a theoretically exhaustive solution to the control problem under the action of parametric and coordinate disturbances that vary over a wide range. Indeed, for a system with deviation control, the block diagram of which is shown in Fig. 3.13, transfer function along the disturbance channel

$$\Phi(p)=\frac{y(p)}{f(p)}=\frac{W_2(p)}{1+K_k(p)W_1(p)W_2(p)}, \qquad (3.509)$$

tends to zero at $K_k(p)\to\infty$.

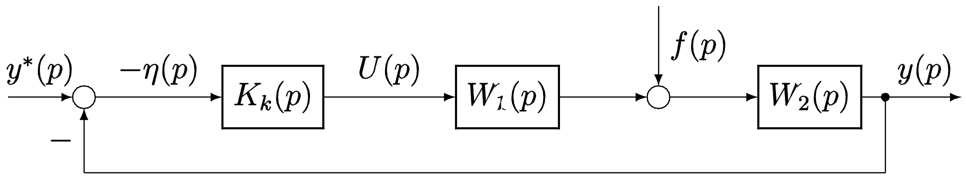

Fig. 3.13. To the assessment of static properties

Transfer function with respect to the reference input

$$\Phi(p)=\frac{y(p)}{y^*(p)}=\frac{K_k(p)W_1(p)W_2(p)}{1+K_k(p)W_1(p)W_2(p)} \qquad (3.510)$$

as $K_k(p)\to\infty$ tends to one. Consequently, the desired static properties of the relay system can be achieved by approximating the characteristics of its real sliding mode closer to the characteristics of the ideal one and thus realizing a sufficient gain of the controller.

Let us now consider in the most general formulation the problem of the invariance of systems with an infinite gain to changes in the parameters of the control object. For the generalized block diagram shown in Fig. 3.14, the following equations can be written:

$$x(p)=z(p)-z_1(p)=z(p)-W_0(p)U(p);$$
$$z(p)=y^*(p)-y(p)=y^*(p)-W_1(p)U(p). \qquad (3.511)$$

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In sliding mode, the signal at the input of the relay element is zero, therefore $z(p) - W_0(p)U(p) = 0$, whence

$$U(p) = \frac{1}{W_0(p)} z(p). \tag{3.512}$$

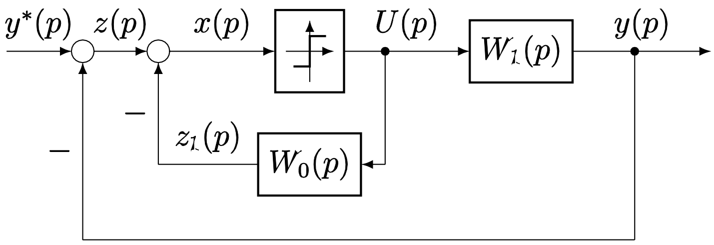

Fig. 3.14 . To the justification of invariance

The variable $U(p)$ represents some equivalent control input. The equation that determines the movement of the relay system in the sliding mode is obtained by substituting the expression (3.38) into the equation (3.37)

$$z(p) = \frac{W_0(p)}{W_0(p) + W_1(p)} y^*(p). \tag{3.513}$$

This equation corresponds to the system, the block diagram of which is shown in Fig. 3.14, if we replace the relay element with a linear amplifier with an infinitely large gain. The equation (3.39) is also valid for the system, the block diagram of which is shown in Fig. 3.15, leading to the conclusion that the movement of a relay system operating in a sliding mode is not influenced by the parameters of the linear part of the control object, covered by feedback together with the relay element.

The above considerations show that a relay system in a sliding mode, which belongs to the class of systems that are stable with an unlimited increase in the gain, has the property of invariance with respect to certain parametric and coordinate disturbances.

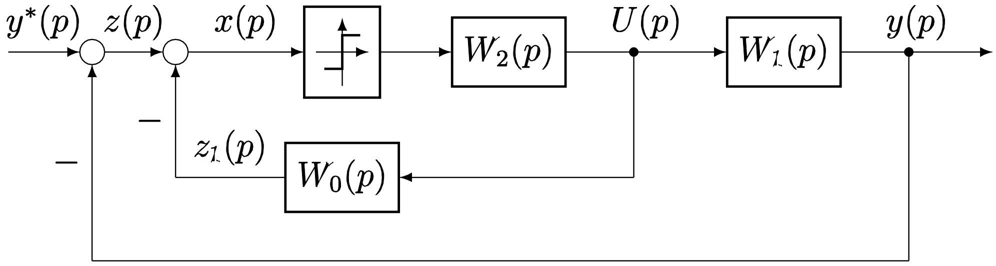

Fig. 3.15. Equivalent block diagram

Introducing several relays into the control system and creating a sliding mode for each of them by covering feedback jointly with each element of the linear part of the control object makes it possible to eliminate the influence of almost all its variable parameters and compensate for the influence of external disturbances on the dynamic properties of the system.

3.1.4. Real sliding mode

In real systems, switching of a relay element does not occur on the line described by the equation $S(\eta) = 0$, but in some vicinity of it $S(\eta) \pm A(\Delta)$ (Fig. 3.16).

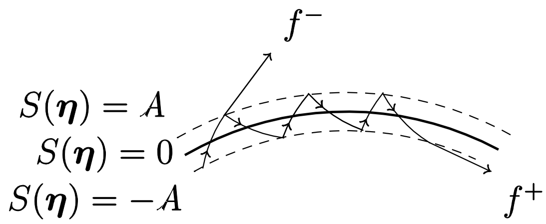

*Fig. 3.16. S*witching of the relay element

The radius of this vicinity depends on the parameter Δ, which determines switching imperfections – spatial or temporal delay, dynamics of fast unmodeled elements of the control system, etc.

As the parameter Δ approaches zero, the amplitude and frequency of the sliding mode will satisfy the limits

$$\lim_{\Delta\to 0}\omega(\Delta)=\infty;\ \lim_{\Delta\to 0}A(\Delta)=0, \tag{3.40}$$

and real sliding turns into ideal sliding.

In addition to the amplitude $A(\Delta)$, the real sliding mode is characterized by the switching frequency, which can be estimated in accordance with the following expression

$$\omega(\Delta)\le\frac{\pi\,\|f\|}{2A(\Delta)}, \tag{3.514}$$

where $\|f\|=\min(\|f^-\|,\|f^+\|)$.

The equations for the existence of the sliding mode (3.10) are supplemented by the inequality

$$|S(\boldsymbol{\eta})|\le A(\Delta), \tag{3.515}$$

and for the existence of a real sliding mode the condition

$$\operatorname{grad}S(\boldsymbol{\eta})\times\mathbf{f}^+(\boldsymbol{\eta},U)<0 \tag{3.516}$$

when the coordinate of the representing point on the switching line $S(\boldsymbol{\eta})$ reaches the value $A(\Delta)$.

Similar condition

$$\operatorname{grad}S(\boldsymbol{\eta})\times\mathbf{f}^-(\boldsymbol{\eta},U)>0 \tag{3.517}$$

must be fulfilled when $S(\boldsymbol{\eta})>-A(\Delta)$.

Thus, in real sliding modes, the conditions of existence are “smeared” in space and are considered not on a line, but on a hyperplane.

Drawing up equations for the dynamics of a closed-loop system in a real sliding mode is complicated by the fact that the vector $\mathbf{f}^0_{\text{re}}(\boldsymbol{\eta})$ does not lie on the tangent to the switching line.

Therefore, for an affine dynamic object, whose perturbed motion equations take the form

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$$p\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) + \mathbf{M}u \tag{3.518}$$

the speed of the representing point in the sliding mode is determined by the expression

$$pS(\boldsymbol{\eta}) = \nabla S(\boldsymbol{\eta})\mathbf{f}(\boldsymbol{\eta}) + \nabla S(\boldsymbol{\eta})\mathbf{M}u. \tag{3.519}$$

If $\nabla S(\boldsymbol{\eta})\mathbf{M} \neq 0$, then the control input can be found from the equation (3.45)

$$u = -\frac{\nabla S(\boldsymbol{\eta})\mathbf{f}(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})\mathbf{M}} + \frac{pS(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})\mathbf{M}}. \tag{3.520}$$

Substituting the control input (3.46) into the equations (3.44) allows us to determine the motion equations of a closed-loop system in real sliding mode

$$p\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) - \mathbf{M} \times \left(\frac{\nabla S(\boldsymbol{\eta})\mathbf{f}(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})\mathbf{M}} + \frac{pS(\boldsymbol{\eta})}{\nabla S(\boldsymbol{\eta})\mathbf{M}} \right). \tag{3.521}$$

Unlike the ideal sliding mode, in the real sliding mode the components of the system state vector continuously change and therefore the right side of the equation (3.47) does not turn into zero. This leads to the fact that the coordinate of the representing point on the switching line equals zero only at certain moments in time and therefore switching of the relay element occurs with a finite frequency, introducing disturbances into the control process. These disturbances can be eliminated by increasing the sliding order.

3.2. High-order sliding modes

3.2.1. Conditions of emergence and existence

In addition to amplitude and frequency, the sliding mode is characterized by the sliding order. Let us examine this concept in more detail. We will assume that the perturbed motion of the control object is described by the system (3.44) and is compensated by ideal discontinuous control

$$u = -\operatorname{sign}(S(\boldsymbol{\eta})). \tag{3.522}$$

Substituting the control input (3.48) into the equation (3.45), we get an expression for the motion speed along the switching line

$$pS(\boldsymbol{\eta}) = \nabla S(\boldsymbol{\eta})\mathbf{f}(\boldsymbol{\eta}) - \nabla S(\boldsymbol{\eta})\mathbf{M}\operatorname{sign}(S(\boldsymbol{\eta})). \tag{3.523}$$

The equation (3.49) shows how the motion speed along the switching line changes as a function of the coordinates of the perturbed motion. Moreover, the right-hand side of the equation (3.49) is discontinuous and depends on the sign of the coordinate of the representing point on the switching line $S(\boldsymbol{\eta})$.

A sliding mode for which the derivative of the speed along the switching line $pS(\boldsymbol{\eta})$ has a discontinuity at $S(\boldsymbol{\eta}) = 0$ will be called *an ideal first-order sliding mode*. Similarly, a real first-order sliding mode is characterized by discontinuity of the derivative $pS(\boldsymbol{\eta})$ when the representing point hits the vicinity $A(\Delta)$ of the switching line.

If the conditions $S(\boldsymbol{\eta}) = 0,\ pS(\boldsymbol{\eta}) = 0$ are met, and both of these functions are continuous and there is a discontinuity in the second derivative $p^2S(\boldsymbol{\eta})$, a second-order sliding mode occurs. Unlike the first-order sliding mode, which is "flat" and occurs when the representing point hits the switching hyperplane, the second-order sliding mode is "spatial" and occurs in space, formed by two hyperplanes $S(\boldsymbol{\eta}) = 0$ and $pS(\boldsymbol{\eta}) = 0$ (Fig. 3.17–3.18).

Generalizing the provided calculations, we can assert that for the emergence of a sliding mode of the n-th order, it is necessary that the switching function itself and $n-1$ of its derivatives in the vicinity of the origin be continuous and equal to zero, and the derivative of the n-th order must have a discontinuity at the origin.

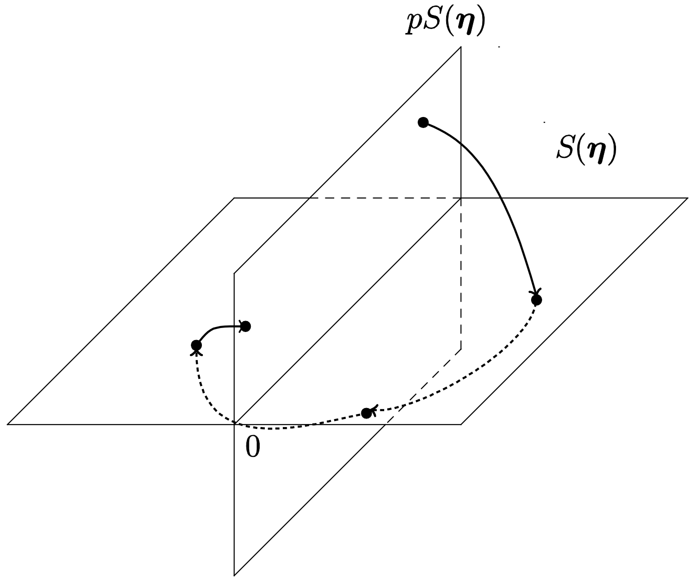

Fig. 3.17. Switching space

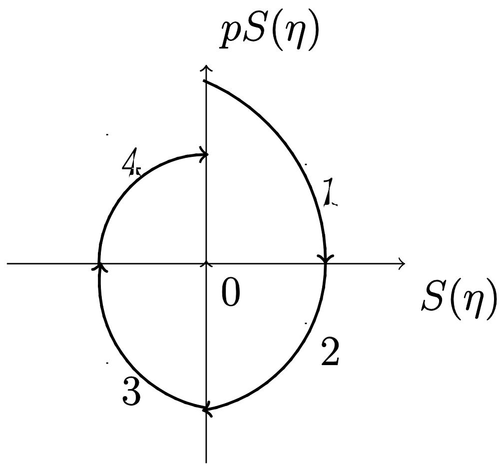

Fig. 3.18. Projection onto the plane (S, pS)

The order of the sliding mode can be associated with the order of the control object if the output is taken as the variable $S(\mathbf{\eta})$.

Let us demonstrate this using the example of a linear object, whose transfer function is the ratio of the m-th and n-th order polynomials

$$W(p) = \frac{S(\mathbf{\eta})}{u} = \frac{P(p)}{Q(p)}. \tag{3.524}$$

Then, with a relative order of the system equal to one, the derivative $pS(\mathbf{\eta})$ will be proportional to the control input and will be discontinuous if the control input itself is discontinuous. Thus, the number of the derivative that clearly depends on the control input, determines the order of the sliding mode. However, in this case it is necessary to verify the conditions for the occurrence of sliding and hitting the switching line, which in the case of a sliding mode of arbitrary order r can be generalized as follows:

• conditions for the occurrence of sliding

$$p^r S^+(\boldsymbol{\eta}) = \underbrace{\nabla \cdots}_{r-1}\nabla S(\boldsymbol{\eta}) \times \mathbf{f}^+(\boldsymbol{\eta},U)\underbrace{\cdots \times \mathbf{f}^+(\boldsymbol{\eta},U)}_{r-1} < 0;$$
$$p^r S^-(\boldsymbol{\eta}) = \underbrace{\nabla \cdots}_{r-1}\nabla S(\boldsymbol{\eta}) \times \mathbf{f}^-(\boldsymbol{\eta},U)\underbrace{\cdots \times \mathbf{f}^+(\boldsymbol{\eta},U)}_{r-1} > 0. \quad (3.525)$$

or

$$p^r S^+(\boldsymbol{\eta}) = p(p^{r-1} S^+(\boldsymbol{\eta})) < 0,\ r = 2,\cdots,n;$$
$$pS^+(\boldsymbol{\eta}) = \nabla S(\boldsymbol{\eta}) \times \mathbf{f}^+(\boldsymbol{\eta},U);$$
$$p^r S^-(\boldsymbol{\eta}) = p^{r-1} S^-(\boldsymbol{\eta}) > 0,\ r = 2,\cdots,n;$$
$$pS^-(\boldsymbol{\eta}) = \nabla S(\boldsymbol{\eta}) \times \mathbf{f}^-(\boldsymbol{\eta},U). \quad (3.526)$$

• conditions for hitting the switching line

$$\prod_{i=0}^{r} p^i S(\boldsymbol{\eta}) < 0. \quad (3.527)$$

3.2.2. Principles of construction of high-order sliding algorithms

Let us consider the perturbed motion of a second-order dynamic system, which is described by differential equations in the Brunovsky form

$$p\eta_1 = \eta_2;$$
$$p\eta_2 = u. \quad (3.528)$$

Stable motion of the system from an arbitrary position to the origin, which occurs under the influence of control u, can occur along monotonic (Fig. 3.19), aperiodic (Fig. 3.20) and oscillatory (Fig. 3.21) trajectories.

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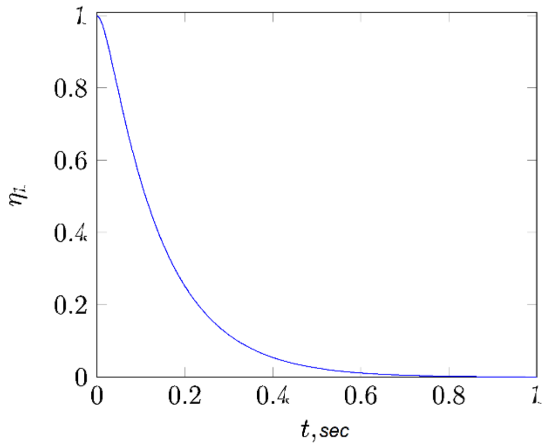

Fig. 3.19. Monotonic trajectory of a second-order object

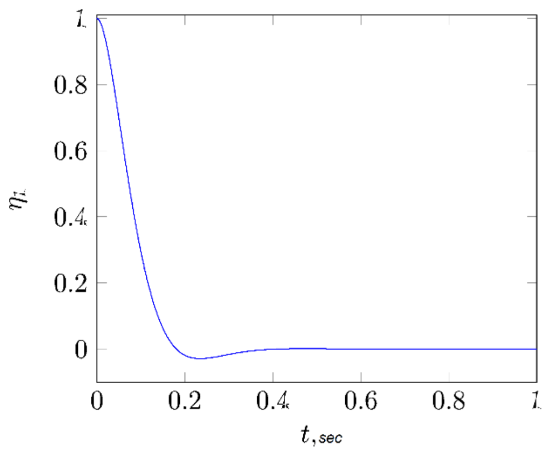

Fig. 3.20. Aperiodic trajectory of a second-order object

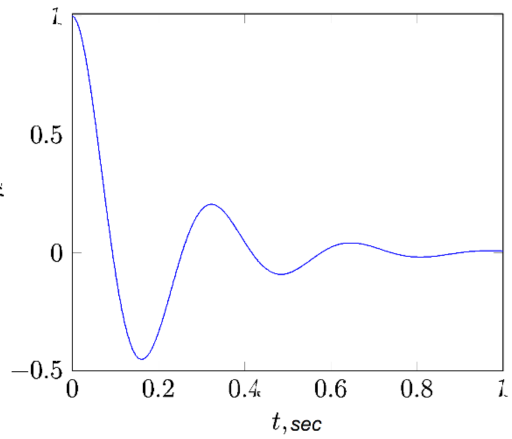

Fig. 3.21. Oscillatory trajectory of a second-order object

The corresponding sliding trajectories are shown in Fig. 3.22 – 3.24.

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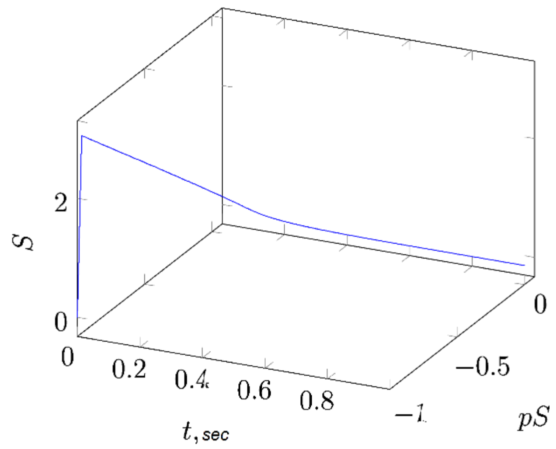

Fig. 3.22. Monotonic sliding trajectory of the second-order

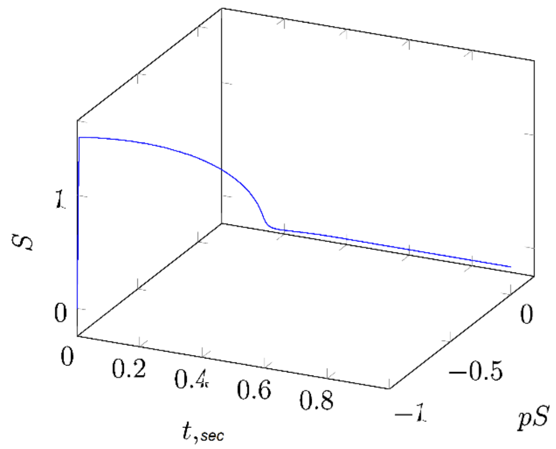

Fig. 3.23. Aperiodic sliding trajectory of the second-order

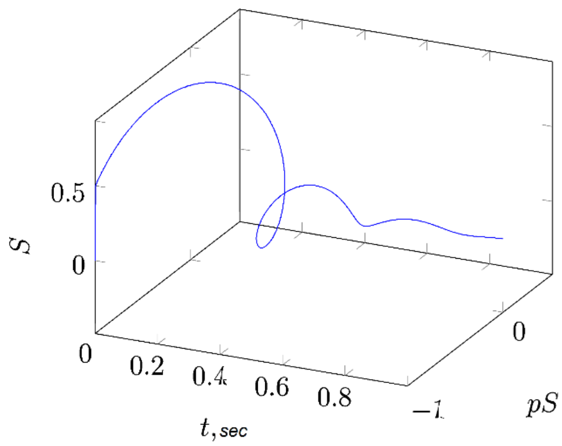

Fig. 3.24. Oscillatory sliding trajectory of the second-order

The nature of transient processes shown in Fig. 3.22 – 3.24 is explained by the unique properties and parameters of closed-loop classical discontinuous control systems.

Let us consider their possible modifications, in which high-order sliding modes arise.

Let a control input applied to the object (3.54) be

$$u = -k_1 \operatorname{sign} \eta_1 - k_2 \eta_2. \tag{3.529}$$

Then the equations of the perturbed motion of the closed-loop control system will take the form

$$p\eta_1 = \eta_2; \quad p\eta_2 = -k_1 \operatorname{sign} \eta_1 - k_2 \eta_2. \tag{3.530}$$

Let us show that the control input (3.55) ensures the occurrence of a second-order sliding mode with respect to the variable η_2. To do this, we rewrite the control input (3.55) as follows

$$u = -k_1 \operatorname{sign} S_1 - S_2, \tag{3.531}$$

Where $S_1 = \eta_1, S_2 = k_2 \eta_2$.

It is evident that at $\eta_1 = 0$ the function S_1 takes zero value, and the derivative pS_1 due to the second equation of the system (3.56) is

$$pS_1 = p\eta_1 = \eta_2. \tag{3.532}$$

The first derivative pS_1 is also zero at the origin of the phase plane.

Let us now find the second derivative of the function S_1.

$$p^2 S_1 = p^2 \eta_1 = p\eta_2 = -k_1 \operatorname{sign} \eta_1 - k_2 \eta_2. \tag{3.533}$$

The function S_1 and its first derivative are continuous, while its second derivative $p^2 S_1$ is discontinuous. These factors create the prerequisites for the emergence of a second-order sliding mode in the considered system.

We will now demonstrate that the motion of system (3.56) is stable. To do this, we write the positive definite Lyapunov function

$$V = V_{11} \mid \eta_1 \mid + V_{22} \eta_2^2, \tag{3.534}$$

whose derivative is

$$pV = \frac{\partial V}{\partial \eta_1} p\eta_1 + \frac{\partial V}{\partial \eta_2} p\eta_2 =$$
$$= V_{11} sign\eta_1 \times \eta_2 + 2V_{22}\eta_2 \times (-k_1 sign\eta_1 - k_2\eta_2) =$$
$$= (V_{11} - 2V_{22}k_1) sign\eta_1 \times \eta_2 - 2V_{22}k_2\eta_2^2. \tag{3.535}$$

At

$$V_{11} = 2V_{22}k_1, \tag{3.536}$$

the first term of the derivative (3.61) becomes zero, and the derivative itself becomes negatively definite

$$pV = -2V_{22}k_2\eta_2^2. \tag{3.537}$$

Consequently, according to Lyapunov's theorem on the stability of motion, the considered system is asymptotically stable.

Now let us consider the case when the control input provides sliding in multiple switching planes.

Let the control input applied to the object (3.54) have the form

$$u = -k_1 |\eta_1| \text{sign}(\xi), \tag{3.538}$$

where

$$\xi = \eta_1 + k_2\eta_2. \tag{3.539}$$

For the new variable ξ, sliding mode in a closed-loop system

$$p\eta_1 = \eta_2; \quad p\eta_2 = -k_1 |\eta_1| \text{sign}(\xi) \tag{3.540}$$

is a first-order sliding mode.

However, in the system

$$p\eta_1 = \eta_2; \quad p\eta_2 = -k_1 |\eta_1| \text{sign}(\eta_1 + k_2\eta_2) \tag{3.541}$$

a second-order sliding mode arises with respect to the variable η_1. This is due to the fact that the function $S \equiv \xi$ is uniquely zero at the origin, while its derivative

$$pS = p\xi = p(k_1|\eta_1|\operatorname{sign}(\eta_1 + k_2\eta_2)) =$$
$$= k_1 \operatorname{sign}(\eta_1) p\eta_1 \operatorname{sign}(\eta_1 + k_2\eta_2) + k_1|\eta_1|\delta(\eta_1 + k_2\eta_2), \quad (3.542)$$

where $\delta(\eta_1 + k_2\eta_2)$ is the Dirac function and it is discontinuous.

Summarizing provided calculations, we can assert that in order to form a second-order sliding mode, the control input must be a combination of continuous and discontinuous functions in various combinations.

3.3. Second-order sliding algorithms

Currently, there are three groups of control algorithms that implement second-order sliding modes:

- Twisting algorithm;
- Nested algorithm;
- Super-twisting algorithm.

Let us examine these algorithms in more detail.

3.3.1. Twisting algorithm

The twisting algorithm is implemented by a control input of the form

$$u = -U_1 \operatorname{sign}(\eta_1) - U_2 \operatorname{sign}(p\eta_1), \quad (3.543)$$

where $U_1 > U_2 > 0$.

The closed-loop system is described by the equations

$$p\eta_1 = \eta_2; \quad p\eta_2 = -U_1 \operatorname{sign}(\eta_1) - U_2 \operatorname{sign}(p\eta_1), \quad (3.544)$$

and the corresponding phase trajectories are segments of parabolas (Fig. 3.25).

Let t_0,..., t_4 be consecutive moments of switching time of the control input, and the time intervals between switches are determined by the relationship

$$\tau_i = t_i - t_{i-1}. \tag{3.545}$$

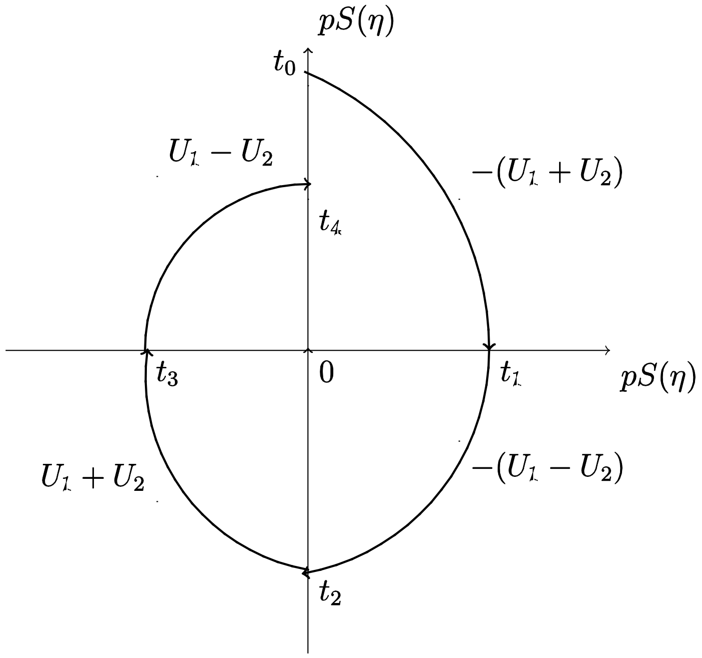

Fig. 3.25. Switching of the relay element

The relationships between consecutive values of variables η_1 and η_2 at switching moments are governed by the following equations:

• for 1st and 3rd quadrants

$$\eta_1 = \pm \frac{\eta_2^2}{U_1 + U_2}; \tag{3.546}$$

• for 2nd and 4th quadrants

$$\eta_1 = \pm \frac{\eta_2^2}{U_1 - U_2}. \tag{3.547}$$

Therefore, the coordinates of the perturbed motion in the upper and lower coordinate half-planes of the phase plane at the moments t_0 and t_2 are governed by the relationship

$$\eta_2^2(t_2) = \frac{U_1 - U_2}{U_1 + U_2} \eta_2^2(t_0) = U\eta_2^2(t_0), \tag{3.548}$$

where

$$U = \frac{U_1 - U_2}{U_1 + U_2}. \quad (3.549)$$

Similarly, for moments t_2 and t_4, we have

$$\eta_2^2(t_4) = U\eta_2^2(t_2) \quad (3.550)$$

or

$$\eta_2^2(t_4) = U^2\eta_2^2(t_0).$$

(3.551)

Thus, for the i-th complete revolution of the phase vector around the origin, the following relations between the state variables of the system at different times are true:

$$\eta_2^2(i) = U^2\eta_2^2(i-1) = \cdots = U^{2i}\eta_2^2(0), i = 1, 2, \cdots \quad (3.552)$$

or after extracting the square root and taking the absolute values

$$|\eta_2(i)| = U|\eta_2(i-1)| = \cdots = U^i|\eta_2(0)|, i = 1, 2, \cdots. \quad (3.553)$$

It is obvious that the series $(1 \quad U \quad \cdots \quad U^i)$, formed by the coefficients of the expression (3.79), converges if $U < 1$. This condition is automatically satisfied by appropriately choosing the values of U_1 and U_2.

Thus, any motion trajectory that begins at an arbitrary point in the phase space reaches the origin, sequentially passing through all quadrants of the phase plane, as if "twisting."

3.3.2. Nesting Algorithm

In the considered twisting algorithm, feedbacks on state variables are linear. If the feedback on the controlled variable becomes nonlinear, a nested control algorithm can be obtained

$$u = -U_{\max}\,\mathrm{sign}[\xi(\eta_1, \eta_2)], \quad (3.554)$$

where

$$\xi(\eta_1,\eta_2) = -f(\eta_1) + \eta_2. \qquad (3.555)$$

The function $f(\eta_1)$ in the equation (3.81) is a smooth scalar function that vanishes at the origin. An example of such a function is an irrational dependence of the form

$$f(\eta_1) = -k_1 |\eta_1|^{\alpha} \operatorname{sign}(\eta_1). \qquad (3.556)$$

With sufficient control energy reserve, the equation $k_1 |\eta_1|^{\alpha} \operatorname{sign}(\eta_1) + \eta_2 = 0$

(3.557)

is the sliding curve of a closed-loop system

$$p\eta_1 = \eta_2; \quad p\eta_2 = -U_{\max} \operatorname{sign}[k_1 |\eta_1|^{\alpha} \operatorname{sign}(\eta_1) + \eta_2]. \qquad (3.558)$$

Reaching this curve guarantees the transfer of the control system from any point in the phase space to the origin in a finite time (Fig. 3.26).

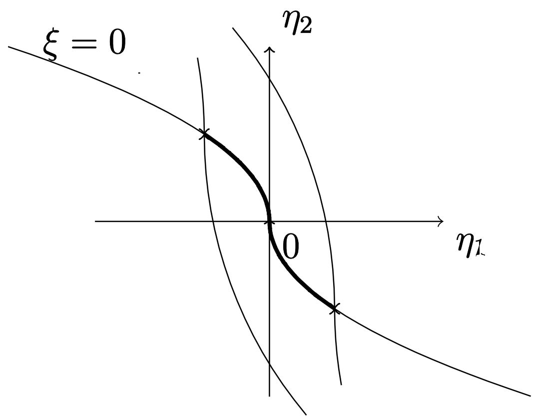

Fig. 3.26. Phase plane

3.3.3. Super-twisting algorithm

Using the super- twisting algorithm

$$u = -U_1 \operatorname{sign} \eta_1 - \frac{U_2}{|\eta_1|^{0,5}} p\eta_1, U_1 > 0, U_2 > 0 \qquad (3.559)$$

leads to the emergence of a second-order sliding mode in the dynamic system (3.54).

In this case, the motion equations of the closed-loop system have the form

$$p\eta_1 = \eta_2; \quad p\eta_2 = -U_1 \operatorname{sign} \eta_1 - \frac{U_2}{|\eta_1|^{0,5}} p\eta_1. \tag{3.560}$$

Let us represent the system of two equations (3.86) in the form of a second-order equation

$$p^2\eta_1 + \frac{U_2}{|\eta_1|^{0,5}} p\eta_1 + U_1 \operatorname{sign} \eta_1 = 0. \tag{3.561}$$

A distinguishing feature of the dynamic system, whose perturbed motion is described by the differential equation (3.87), is a variable damping coefficient, which increases indefinitely as it approaches the origin.

When the sign of the deviation η_1 changes, the system's motion trajectories do not change qualitatively, since the form of the equation remains the same. This allows us to consider following equation instead of the equation (3.87)

$$p^2\eta_1 + \frac{U_2}{|\eta_1|^{0,5}} p\eta_1 + U_1 = 0. \tag{3.562}$$

Qualitative trajectories of the system are shown in Fig. 3.27.

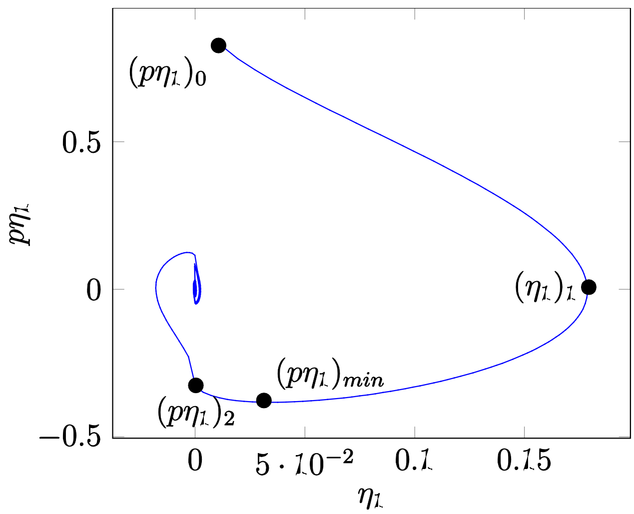

Fig. 3.27. Phase trajectory of the system

Fig. 3.27 indicates the characteristic points of the system operation that will be used in further calculations: $(p\eta_1)_0$ – the initial value of the system speed, $(\eta_1)_1$ – the maximum deviation, $(p\eta_1)_{\min}$ – the minimum value of the system speed, $(p\eta_1)_2$ – the second intersection of the phase trajectory of the ordinate axis.

Let us transform the equation (3.88) as follows

$$p\left[p\eta_1 + 2U_2 p\sqrt{\eta_1} + U_1 / p\right] = 0. \quad (3.563)$$

Integrating the equation (3.89) allows us to write down the dynamics of the system, governed by the equation (3.88) as

$$p\eta_1 + 2U_2 p\sqrt{\eta_1} + U_1 / p = (p\eta_1)_0. \quad (3.564)$$

At maximum deviation $(\eta_1)_1$, the speed of the system $(p\eta_1)_1$ is zero. Let us denote this moment of time as t_1 and write the equation (3.90) for it

$$2U_2 p\sqrt{\eta_1} + U_1 t_1 = (p\eta_1)_0. \quad (3.565)$$

Since $t_1 > 0$, then the following inequality turns out to be true

$$\sqrt{\eta_1} < \frac{(p\eta_1)_0}{2pU_2}. \quad (3.566)$$

The equation (3.91) together with the inequality (3.92) shows that the deviation η_1 decreases over time. In addition, time t_1 can be found from the equation (3.91). Assuming that $\eta_1 = 0$, we can rewrite the equation (3.91) as follows

$$U_1 t_1 = (p\eta_1)_0, \quad (3.567)$$

where

$$t_1 = \frac{(p\eta_1)_0}{U_1}. \quad (3.568)$$

Now let us consider the moment t_2 when the speed of the system reaches its minimum value $(p\eta_1)_{\min}$. It is obvious that at this moment $(p^2\eta_1) = 0$, which means the equation (3.89) will take the form

$$\frac{U_2}{\eta_1((p\eta_1)_{\min})^{0,5}}(p\eta_1)_{\min}+U_1=0 \tag{3.569}$$

or

$$U_2(p\eta_1)_{\min}+U_1\eta_1((p\eta_1)_{\min})^{0,5}=0. \tag{3.570}$$

From the equation (3.96), we can find the coordinate η_1 at which the speed of movement will be minimal

$$\eta_1((p\eta_1)_{\min})^{0,5}=-\frac{U_2}{U_1}(p\eta_1)_{\min}. \tag{3.571}$$

From Fig. 3.27 it is clear that $\eta_1>\eta_1((p\eta_1)_{\min})$ and $(p\eta_1)_2>(p\eta_1)_{\min}$, so the following inequalities are true

$$\sqrt{(\eta_1)_1}>\sqrt{\eta_1((p\eta_1)_{min})};\quad -\frac{U_2}{U_1}(p\eta_1)_{min}>-\frac{U_2}{U_1}(p\eta_1)_2. \tag{3.572}$$

$$\sqrt{(\eta_1)_1}\,\frac{U_1}{U_2}>-(p\eta_1)_2. \tag{3.573}$$

Substituting the inequality (3.92) into (3.99), we get the relationship between the speeds of two consecutive intersections of the ordinate axis of the phase plane

$$\frac{(p\eta_1)_0}{2U_2}\frac{U_1}{U_2}>-(p\eta_1)_2 \tag{3.574}$$

or

$$\frac{(p\eta_1)_2}{(p\eta_1)_0}<\frac{U_1}{2U_2^2}. \tag{3.575}$$

Denoting the right-hand side of the inequality (3.101) as

$$\gamma=\frac{U_1}{2U_2^2}, \tag{3.576}$$

we can generalize the result for any two consecutive intersections of the ordinate axis

$$\left|\frac{(p\eta_1)_{i+1}}{(p\eta_1)_i}\right| \le \gamma. \tag{3.577}$$

The inequality (3.103) shows that if $\gamma < 1$, then the transition process will be stable.

3.3.4. Modification of the super-twisting algorithm

The previously discussed algorithms ensure stable perturbed motion from any starting point to the origin. However, such motion is oscillatory, which is explained by the fact that the corresponding closed-loop systems have a relative order of 2. Therefore, the problem arises of ensuring the asymptotic stability of closed-loop systems in which second-order sliding modes occur.

We will look for a solution to this problem within the class of super-twisting algorithms. One of the varieties of algorithms in this class is the algorithm [37]

$$u = -U_1 |\eta|^{0,5} \operatorname{sign}\eta - U_2 \operatorname{sign}\eta, \tag{3.578}$$

used in first-order dynamic object control systems

$$p\eta = a_{11}\eta + mu. \tag{3.579}$$

Substituting the control input (3.104) into the equation (3.105) allows us to write the motion equation of the closed-loop system

$$p\eta = a_{11}\eta - mU_1 |\eta|^{0,5} \operatorname{sign}\eta - mU_2 \operatorname{sign}\eta. \tag{3.580}$$

Since the closed-loop system (3.106) is a first-order system, the processes occurring in it are asymptotically stable, and in the sliding mode the motion occurs along the switching line

$$\eta = 0. \tag{3.581}$$

Let's factor out the component $\operatorname{sign}\eta$ in the algorithm (3.104)

$$u = -\left(U_1 |\eta|^{0,5} + U_2\right) \operatorname{sign}\eta. \tag{3.582}$$

The resulting algorithm can be considered as a standard discontinuous control algorithm, but with a variable amplitude, which can vary from the value $\sqrt{2}U_1 + U_2$, when the motion of the system starts from one extreme position and must end in the other extreme, to the value U_2 when the object's deviation from the desired position is compensated. Thus, when the system reaches the equilibrium line, switching of the control input with amplitude U_2 and high frequency will be observed. In some cases, a long-term high-frequency switching regime can lead to damage to the control object. Therefore, along with the algorithm (3.108), we will consider the algorithm obtained from (3.108) when the component U_2 approaches zero

$$u = -U_1 |\eta|^{0,5} |\operatorname{sign}\eta. \quad (3.583)$$

The difference between the control input (3.108) (Fig. 3.28) and (3.109) (Fig. 3.29) is the continuity of the latter.

Let us show that the resulting algorithm also allows to implement a second-order sliding mode.

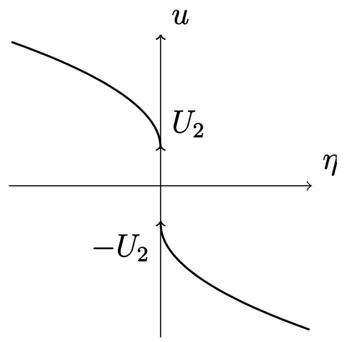

Fig. 3.28. Static characteristics of the controller with control algorithm (3.108)

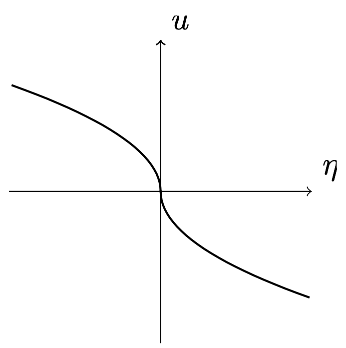

Fig. 3.29. Static characteristics of the controller with control algorithm (3.109)

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We will assume that $S = \eta$, then at the origin

$$\eta = 0; S = 0;$$
$$pS = p\eta = u = -U_1 |\eta|^{0.5}\ sign\eta = 0;$$
$$p^2S = pu = -U_1\left(\frac{p|\eta|}{2|\eta|^{0.5}}sign\eta - |\eta|^{0.5}|\delta(\eta)\right) =$$
$$= -U_1\left(\frac{sign\eta \times p|\eta|}{2|\eta|^{0.5}}sign\eta - |\eta|^{0.5}|\delta(\eta)\right) =$$
$$= -0.5U_1^2 sign\eta - U_1|\eta|^{0.5}|\delta(\eta). \qquad (3.584)$$

Since both terms of the second derivative p^2S are discontinuous, it can be argued that a second-order sliding mode appears in the system implementing the algorithm (3.109).

We will modify algorithms (3.108) and (3.109) in such a way that they can be used to control objects of arbitrary order. To do this, we replace the switching line equation (3.107) with the generalized one

$$S = \sum_{i=1}^{n} k_i\eta_i = 0, \qquad (3.585)$$

where k_i are the weighting coefficients that ensure the achievement of the predetermined control goal.

Then, taking into account the equation (3.111), the algorithm (3.108) can be rewritten as

$$u = -\left(U_1|S|^{0,5} + U_2\right)\operatorname{sign} S \qquad (3.586)$$

or in expanded form

$$u = -\left(U_1\left|\sum_{i=1}^{n} k_i\eta_i\right|^{0,5} + U_2\right)\operatorname{sign}\left(\sum_{i=1}^{n} k_i\eta_i\right). \qquad (3.587)$$

For the considered second-order object, the algorithm (3.113) can be represented as follows

$$u=-\left(U_1\,|k_1\eta_1+k_2\eta_2|^{0,5}+U_2\right)\mathrm{sign}(k_1\eta_1+k_2\eta_2)\,. \qquad (3.588)$$

Similarly, the algorithm (3.109) in general form will be

$$u=-U_1\left|\sum_{i=1}^{n}k_i\eta_i\right|^{0,5}\mathrm{sign}\left(\sum_{i=1}^{n}k_i\eta_i\right), \qquad (3.589)$$

and for the considered object

$$u=-U_1\left|k_1\eta_1+k_2\eta_2\right|^{0,5}\mathrm{sign}(k_1\eta_1+k_2\eta_2). \qquad (3.590)$$

A distinctive feature of the algorithms (3.113) and (3.115) is the change in the amplitude of the control input as a function of the object's state variables. It can be argued that state variables are activated in accordance with the control algorithm and ensure the formation of the desired level of control input. Therefore, the function preceding the sign term will be called *the activation function*.

Let us consider the concept of "activation function" in more detail.

3.4. Nonlinear activation function

Currently, in the neural networks theory, the concept of "activation function" is widely used to determine the level of neuron activation. Drawing an analogy between artificial neurons [36] (Fig. 3.30) and controllers that use information about the complete state vector of the control system (Fig. 3.31), we can conclude that there structures are identical. In turn, this conclusion allows us to expand the class of functions like "sign" and "limitation" types used to limit the output voltage of the controller by considering nonlinear functions with limited magnitude

$$|\,f(.)\,|\le 1\,, \qquad (3.591)$$

which are used as activation in neural networks.

Therefore, in control theory terms, activation can be defined as a nonlinear function that adjusts the control generated by the controller according to a specified algorithm in accordance with the coordinate value of the representing point.

In the neural network theory, analytical studies of the static and dynamic characteristics of a system with nonlinear activation functions are not performed, which prevents mathematically justified selection of the activation function with subsequent system optimization. Moreover, studies devoted to the optimization of system optimization through the use of high-order sliding modes [37] indicate that functions whose values at the origin are equal to zero and whose derivatives are discontinuous should be used as activation functions.

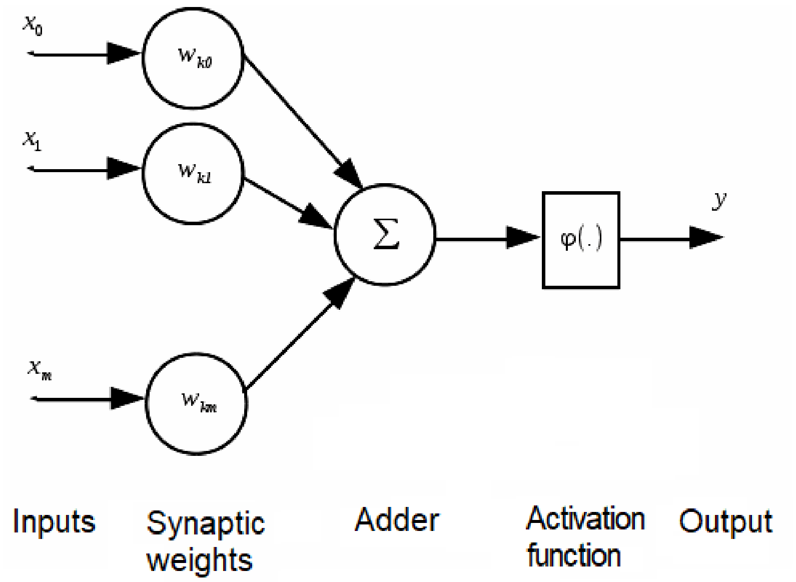

Fig. 3.30. Functional diagram of an artificial neuron

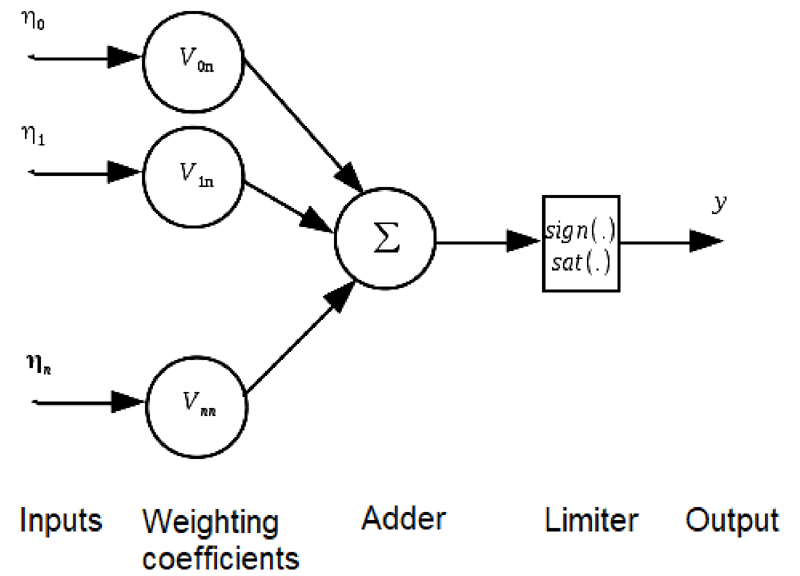

Fig. 3.31. Functional diagram of a controller with compensating feedback

Activation functions that meet such conditions are not widely used in neural networks.

The simplest continuous function with discontinuous derivatives is

$$f(\eta)=\sqrt{\eta}. \tag{3.592}$$

The function (3.118) is defined in the real domain only for positive values of the argument, which does not allow its use in control systems in which signals can have different signs.

Let us generalize the function (3.118) and introduce the nonlinear “square root taking into account the sign” function into consideration

$$\mathrm{sqrt}(\eta)=\sqrt{|\eta|}\,\mathrm{sign}(\eta). \tag{3.593}$$

In the most general case, we will consider a generalization of the function (3.119)

$$\Phi(\eta,\alpha)=|\eta|^{\alpha}\,\mathrm{sign}(\eta),\ \alpha\in[0,1] \tag{3.594}$$

Functions of the form (3.120) will be called *irrational activation functions*. It should be noted that the class of activation functions is not limited only to irrational functions. The activation function can be any function that ensures the achievement of the set goal, gives the desired properties to the closed-loop system, and forms the required motion trajectory.

It is obvious that the functions (3.119) and (3.120) even without taking into account the restrictions (3.117) are nonlinear. In a closed region limited by the condition (3.117), these functions become essentially nonlinear and in the general case are described by the expression

$$\Phi(\eta,\alpha)=\begin{cases}|\eta|^{\alpha}\,\mathrm{sign}(\eta), & if\quad \eta\le 1;\\ \mathrm{sign}(\eta), & if\quad \eta>1.\end{cases} \tag{3.595}$$

3.5. Fractional-order sliding modes

Differentiation of a function (3.120) shows that its derivative

$$\frac{\partial\Phi(\eta,\alpha)}{\partial\eta}=\frac{\partial|\eta|^{a}\operatorname{sign}\eta}{\partial\eta}=|\eta|^{a}\,\delta(\eta)+\alpha\frac{1}{|\eta|^{1-\alpha}} \tag{3.596}$$

consists of two components, each of which at the origin is a discontinuous function. This statement is typical for all irrational activation functions that ensure the occurrence of a high-order sliding mode. However, from the point of view of the conditions for the occurrence of a high-order sliding mode, the presence of two discontinuous functions is redundant and creates basis for clarifying the concept of a high-order sliding mode.

Let us analyze the derivative where the discontinuity of the first term is determined by the discontinuity of the function $\operatorname{sign}(\eta)$ at the origin. The second term becomes discontinuous only when $0\le\alpha\le1$. According to the rules of differentiation of power functions, the order of the derivative is always one unit less than the order of the original function. This is due to the fact that in classical differential calculus only derivatives of integer order are considered.

From the perspective of control theory, this approach is not convenient, since the class of systems with irrational activation functions in which second-order sliding modes are implemented includes both the system with the algorithm

$$u=\lim_{\alpha\to0}|\eta|^{\alpha}\operatorname{sign}\eta, \tag{3.597}$$

which, in its properties, is close to classical discontinuous control systems, and a system with an algorithm

$$u=\lim_{\alpha\to1}|\eta|^{\alpha}\operatorname{sign}\eta, \tag{3.598}$$

whose properties are similar to those of classical linear systems.

Therefore, the concept of high-order sliding modes must be clarified by moving from the concept of an integer sliding order to the concept of a fractional sliding order. Let us use the definition of fractional differentiation operator of power functions [38]

$$D^{a}\eta^{\alpha}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-a+1)}\eta^{\alpha-a}, \tag{3.599}$$

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where $\Gamma(.)$ is the gamma function.

Among all possible orders a of this operator, the case that deserves the most interest is when $a = \alpha$ and

$$D^{\alpha}\eta^{\alpha} = \Gamma(\alpha + 1). \tag{3.600}$$

Unlike classical differential calculus, the fractional derivative of a constant is not equal to zero, so the derivative $D^{\alpha}\,\mathrm{sign}\,\eta$ will be

$$D^{\alpha} sign\eta = k / t^{\alpha} sign\eta, \tag{3.601}$$

here k is a certain coefficient, t is the time from the start of the relay element switching.

Analysis of the derivative (3.127) shows its discontinuity, while the derivative (3.126) is a constant, thus the fractional derivative is

$$\frac{\partial^{\alpha}\Phi(\eta,\alpha)}{\partial^{\alpha}\eta} = \Gamma(\alpha+1) + \frac{k}{t^{\alpha}}\mathrm{sign}\,\eta \; . \tag{3.602}$$

The last expression allows us to state that that the system enters a sliding mode of $1+\alpha$ order.

The above calculations allow us to conclude that the processes occurring in real closed-loop systems with an irrational activation function are quite complex and, unlike similar linear systems, cannot always be described analytically. Therefore, there is a need for analytical research on elementary electromechanical systems with irrational activation functions.

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CHAPTER 4
ANALYTICAL STUDIES OF ELEMENTARY -ELECTROMECHANICAL SYSTEMS WITH IRRATIONAL ACTIVATION FUNCTION

4.1. Open-loop EMS study

Before moving on to the synthesis of electromechanical systems with an irrational function, let us consider how this function affects the characteristics of elementary dynamic systems.

As is known, dynamic systems can be either open-loop or closed-loop. Let us consider a generalized open-loop system with an irrational activation function. Since such a system is nonlinear, the superposition principle is not applicable to its analysis; therefore, it is necessary to distinguish where the nonlinearity is introduced: at the system's input or at its output. (Fig. 4.1 - 4.2)

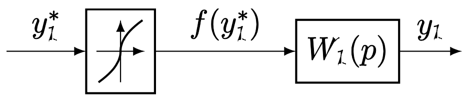

Fig. 4.1. Nonlinearity at the input

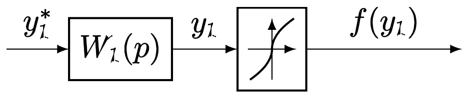

Fig. 4.2. Nonlinearity at the output

To clarify the peculiarities of the trajectories of the systems shown in Fig. 4.1–4.2, we assume that the transfer function $W_1(p)$ corresponds to the first-order aperiodic element. Such a dynamic element describes with a known degree of accuracy the processes occurring in controlled converters, which are an integral part of a closed-loop electromechanical system. Additionally, assuming that the electromagnetic torque formation channel is inertia-free, any electromechanical system (EMS) can be reduced to such an element. Therefore, we will assume that

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$$W_1(p) = \frac{k_1}{T_1 p + 1}, \qquad (4.603)$$

where k_1 is the gain factor of the controlled converter, if processes in a power converter are considered, or the design coefficient of the EMS, if a generalized EMS with an inertia-free electromagnetic channel is being considered; T_1 is the time constant of the converter or EMS, respectively.

Taking the above into account, the block diagrams of the considered systems will take the form shown in Fig. 4.3–4.4.

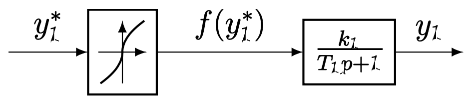

Fig. 4.3. Nonlinearity at the input

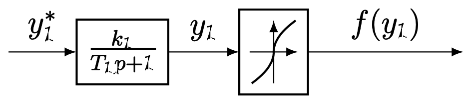

Fig. 4.4. Nonlinearity at the output

4.1.1. Nonlinearity at the system input

The diagram shown in Fig. 4.3 represents the simplest case of a system with an irrational activation function.

$$f(y_1^*) = |y_1^*|^\alpha \operatorname{sign}(y_1^*),\ \alpha \in [0,1],\ |y_1^*| \le 1, \qquad (4.604)$$

which is characterized by the fact that the input signal y_1^* is amplified as a result of raising it to a power then fed into the input of the system. In this case, the equation of motion takes the form

$$T_1 p y_1 + y_1 = k_1 |y_1^*|^\alpha \, sign(y_1^*), \qquad (4.605)$$

where y_1 is the coordinate of the generalized EMS in relative units, y_1^* is the desired value of the EMS coordinate.

By introducing a new control u_1, the equation (4.3) can be represented as follows

$$T_1 p y_1 + y_1 = k_1 u_1. \qquad (4.606)$$

Thus, with a constant signal y_1^*, the nature of the motion of a system with nonlinearity at the input is exponential and does not differ from the motion of a linear EMS.

Despite the identical nature of the transition processes in both linear and nonlinear system, the points of steady-state operation of the EMS will be different when the reference signal changes, resulting in different motion trajectories.

Thus, the static characteristic of a linear system is determined by the expression

$$y_1(\infty) = k_1 y_1^* \qquad (4.607)$$

and linearly depends on the reference signal y_1^*. In the considered nonlinear EMS, the steady-state point is determined by the following expression

$$y_1(\infty) = k_1 |y_1^*|^\alpha \operatorname{sign}(y_1^*) \qquad (4.608)$$

and nonlinearly depends on the reference signal. The limiting case is the constancy of the steady-state point of the EMS operation from the reference signal y_1^* as the exponent α approaches to zero.

$$\lim_{\alpha \to 0} y_1(\infty) = k_1. \qquad (4.609)$$

The obtained dependencies are constructed for a first-order aperiodic element, under the condition that nonlinearity is present at its input. However, as was shown earlier, by selecting the appropriate control input, the motion equations of any electric drive can be represented in the controlled Brunovsky form, which corresponds to the zeroing of the roots of the characteristic equation of the EMS.

Thus, in the case when the system dynamics is described by the equation (4.3), it can be represented as

$$p y_1 = \nu, \qquad (4.610)$$

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where

$$\nu = \frac{k_1}{T_1} | y_1^* |^\alpha \operatorname{sign}(y_1^*) - \frac{1}{T_1} y_1. \tag{4.611}$$

With full compensation of the internal feedback of EMS, the signal at its output increases indefinitely over time, and the movement speed in the nonlinear system

$$p y_1 = \frac{k_1}{T_1} | y_1^* |^\alpha \operatorname{sign}(y_1^*) \tag{4.612}$$

differs from the movement speed of a linear system

$$p y_1 = \frac{k_1}{T_1} y_1^*. \tag{4.613}$$

Considering that the demand signal is limited in magnitude,

$$| y_1^* | < 1, \tag{4.614}$$

the absolute value of the maximum rate of change of the output coordinate will be at $\alpha \to 0$. The minimum speed corresponding to linear EMS is at $\alpha \to 1$.

The obtained result can be generalized to the case of a dynamic object of arbitrary order and it can be stated that the presence of nonlinearity at the input of the system with a constant reference signal y_1^* does not affect the nature of the trajectory of the motion, but affects the steady-state or quasi-steady-state operation mode of the system. Deformations of the motion trajectory, unusual for linear objects and systems, do not occur. When the reference signal changes, the motion trajectories of the nonlinear system differ from the linear one and, due to the amplification of the signal y_1^*, are characterized by large absolute values of the state variables and non-uniformity of the field of steady-state values of the output coordinate of the EMS.

4.1.2. Nonlinearity at the system output

In this case, the dynamics of the first-order nonlinear open-loop EMS is described by a system of equations

$$Tpy_1 + y_1 = k_1 y_1^*;\ y_1' = |y_1|^{\alpha}\ sign(y_1). \tag{4.615}$$

The presence of a nonlinear function $|y_1|^{\alpha}\ \mathrm{sign}(y_1)$ at the output of the system not only changes the steady-state point, which can be determined from the system (4.13) by substituting the first equation into the second at $p = 0$

$$y_1'(\infty) = |k_1 y_1^*|^{\alpha}\ sign(ky_1^*), \tag{4.616}$$

but also distorts the entire trajectory of the electric drive.

Unlike the motion trajectory of a linear system $y_1'(t) = k_1 y_1^*(1 - e^{-t/T_1})$,

(4.617)

the motion trajectory of the considered nonlinear system is described by the equation

$$y_1'(t) = |k_1 y_1^*(1 - e^{-t/T_1})|^{\alpha}\ sign(ky_1^*). \tag{4.618}$$

Thus, the more the exponent α differs from one, the more the trajectory of the nonlinear EMS is distorted compared to the trajectory of the linear system.

In the limit at $\alpha \to 0$, the steady-state point tends to $\mathrm{sign}(ky_1^*)$, and the motion trajectory tends to a unit step function.

A similar picture is observed when EMS is linearized by feedback in order to compensate for internal feedback.

With the exponent $\alpha = 0$, the trajectory of the system

$$y(t) = \frac{k_1}{T_1} y_1^* t \tag{4.619}$$

significantly differs from the motion trajectory at $\alpha \neq 1$

$$y(t) = \left| \frac{k_1}{T_1} y_1^* t \right|^{\alpha} \mathrm{sign}(ky_1^*). \tag{4.620}$$

As follows from the analysis of the expression (4.18), the rate of change of the output variable of the EMS state, where internal feedback is compensated or absent, with nonlinearity at the output differs from a constant even with a constant reference input.

Summarizing the obtained results, the following conclusions can be drawn:

• the introduction of irrational nonlinearity at the output of the EMS leads to a significant changes in the motion trajectory and steady-state point;

• as the exponent α approaches zero, the transition function approximates a step function.

4.2. Mathematical models of closed-loop EMS

The study of elementary closed-loop EMS will be carried out for the structures shown in Fig. 4.5–4.8.

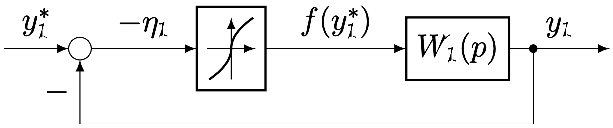

Fig. 4.5. Closed-loop EMS with a nonlinear function in the controller

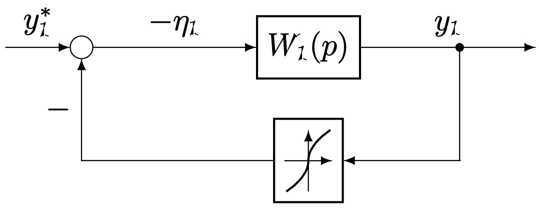

Fig. 4.6. Closed-loop EMS with a nonlinear activation function in the feedback

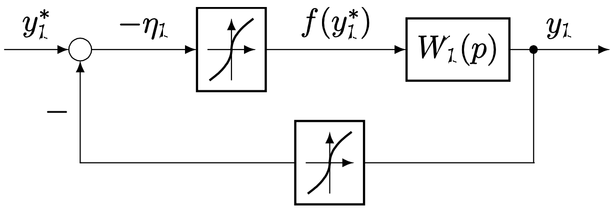

Fig. 4.7. Closed-loop EMS with a nonlinear activation function in the controller and feedback

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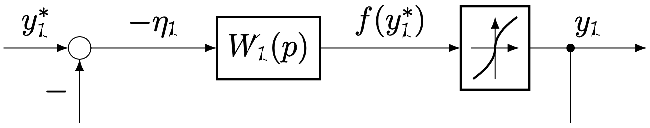

Fig. 4.8 Closed-loop EMS with a nonlinear activation function as part of an electric drive

We will consider a generalized first-order electromechanical object, the dynamics of which is described by a differential equation of the form

$$py_1 = a_{11}y_1 + m_1U_y, \tag{4.621}$$

where $a_{11} = \frac{1}{T}$ is the coefficient accounting for the object's internal feedback, m_1 is the design coefficient, U_y is the control input, y_1 is the controlled coordinate. Having applied the feedback transformation to the object (4.19) and introduced a new control input

$$U = a_{11}y_1 + m_1U_y, \tag{4.622}$$

We can describe its dynamics as follows

$$py_1 = U. \tag{4.623}$$

The implementation of the feedback transformation allowed us to reduce the generalized object (4.19) to an integrating element (4.21). Then the dynamics of the systems shown in Fig. 4.5–4.7 are described by the following equations

• For the scheme shown in Fig. 4.5:

$$py_1 = U; U = -|\eta_1|^{\alpha} \operatorname{sign}(\eta_1), \tag{4.624}$$

where

$$\eta_1 = y_1 - y_1^*, \tag{4.625}$$

here y_1^* is the reference input, y_1 is the controlled coordinate.

Substituting the second equation of the system (4.22) into the first and taking into account the expression (4.23), we get

$$py_1 = -|y_1 - y_1^*|^{\alpha} \operatorname{sign}(y_1 - y_1^*) \tag{4.626}$$

or, assuming that the function $\operatorname{sign}(.)$ can be described by the expression

$$\operatorname{sign}(\eta_1) = \frac{\eta_1}{|\eta_1|}, \tag{4.627}$$

the equation (4.24) will be written as follows

$$py_1 = -|y_1 - y_1^*|^{\alpha} \operatorname{sign}(y_1 - y_1^*) = \\ = -\frac{y_1}{|y_1 - y_1^*|^{\beta}} + \frac{y_1^*}{|y_1 - y_1^*|^{\beta}}, \tag{4.628}$$

where

$$\beta = 1 - \alpha. \tag{4.629}$$

• For the scheme shown in Fig. 4.6:

$$py_1 = U; U = y_1^* - |y_1|^{\alpha} \operatorname{sign}(y_1). \tag{4.630}$$

After substituting the second equation of the system (4.28) into the first and expanding the brackets, we get

$$py_1 = -|y_1|^{\alpha} \operatorname{sign}(y_1) + y_1^*. \tag{4.631}$$

or

$$py_1 = -\frac{y_1}{|y_1|^{\beta}} + y_1^*. \tag{4.632}$$

• For the scheme shown in Fig. 4.7:

$$py_1 = U; \\ U = |y_1^* - |y_1|^{\alpha_1} \operatorname{sign}(y_1)|^{\alpha_2} \operatorname{sign}(y_1^* - |y_1|^{\alpha_1} \operatorname{sign}(y_1)). \tag{4.633}$$

or taking into account the ratio (4.25)

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$$py_1 = \frac{y_1^* - |y_1|^{\alpha_1}\,\mathrm{sign}(y_1)}{\left|y_1^* - |y_1|^{\alpha_1}\,\mathrm{sign}(y_1)\right|^{\beta_2}} = \frac{y_1^* - \dfrac{y_1}{|y_1|^{\beta_1}}}{\left|y_1^* - \dfrac{y_1}{|y_1|^{\beta_1}}\right|^{\beta_2}}. \qquad (4.634)$$

• For the scheme shown in Fig. 4.8:

$$px_1 = U;\ y_1 = |x_1|^{\alpha}\,\mathrm{sign}(x_1);\ U = y_1^* - y_1. \qquad (4.635)$$

Substituting the third equation of the system (4.33) into the first, taking into account the second, allows us to reduce the equation of motion of the closed-loop system to the form

$$px_1 = \left(y_1^* - |x_1|^{\alpha}\,\mathrm{sign}(x_1)\right);\ y_1 = |x_1|^{\alpha}\,\mathrm{sign}(x_1) \qquad (4.636)$$

or taking into account the ratio (4.25)

$$px_1 = \left(y_1^* - \frac{x_1}{|x_1|^{\beta}}\right);\ y_1 = |x_1|^{\alpha}\,\mathrm{sign}(x_1). \qquad (4.637)$$

Expanding the brackets in equation (4.35), we get

$$px_1 = \frac{x_1}{|x_1|^{\beta}} + y_1^*;\ y_1 = |x_1|^{\alpha}\,\mathrm{sign}(x_1). \qquad (4.638)$$

Analysis of expressions (4.26), (4.30) and (4.36) shows that the presence of a nonlinear activation function in a closed-loop system is equivalent to introducing feedback with a variable transfer coefficient. Moreover, in the case where nonlinearity is in front of the object, this coefficient increases as the error decreases. Thus, when the coordinate of the object approaches its desired value, the damping properties of the system are enhanced. A similar phenomenon is observed in classical control systems with a super-twisting algorithm. For dynamic systems in which nonlinearity is at the output of the object or in the feedback, the enhancement of damping properties occurs with a decrease in the state variable of the object y_1.

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4.3 Study of static characteristics of EMS

In the automatic control theory, it is proposed to study static characteristics based on the transfer function of the system, provided that the differentiation operator is zero. This approach is convenient for analyzing linear control systems, since it is part of the complex analysis of closed-loop EMS, which also includes the study of stability, time and frequency characteristics. To perform these studies, the transfer function of the system is certainly necessary, since it allows the use of powerful tools and methods for analyzing dynamic systems. However, for nonlinear systems, compiling a transfer function in a closed-loop state is challenging. Therefore, we will analyze the static properties of systems will be performed based on their equations of their motion in operator form, under the condition that

$$p = 0. \tag{4.639}$$

By equating the left-hand side of the equations (4.26) and (4.30) to zero, we obtain

$$-\frac{y_1}{|y_1 - y_1^*|^{\beta}} + \frac{y_1^*}{|y_1 - y_1^*|^{\beta}} = 0, \tag{4.640}$$

$$-\frac{y_1}{|y_1|^{\beta}} + y_1^* = 0 \tag{4.641}$$

or

$$-y_1 + y_1^* = 0, \tag{4.642}$$

$$-y_1 + y_1^* |y_1|^{\beta} = 0 \tag{4.643}$$

Let us analyze the obtained equations.

• In the case where the control object is an integrating element and the activation function is at its input, in accordance with the equation (4.40), the output variable will reach the desired value in the steady-state. The static characteristic of the system in this case will be

$$y_1 = y_1^*. \tag{4.644}$$

• If the nonlinearity is in the feedback channel, and the steady-state of the system is described by the equation (4.41), then the output and input variables will coincide only at maximum reference inputs $|y_{1\max}^*|=1$. In the general, the static characteristic of such a system is determined by the expression

$$y_1 = \sqrt[1-\beta]{y_1^*} = \sqrt[\alpha]{y_1^*}. \tag{4.645}$$

An analysis of the expression (4.43) shows a linear relationship between the input signal and the output coordinate only in the linear case when $\alpha = 1$. As the exponent α decreases, the nonlinearity of the static characteristic increases. For very small values of the exponent α the static characteristic (4.43) tends to the characteristic of the relay element, and the output coordinate, regardless of the reference input, reaches its maximum value for any values of the setting signal other than zero.

Despite the fact that the considered cases are the simplest in terms of using nonlinear activation functions, it is clear from the expression (4.43) that the coverage of nonlinear feedback even of the integrating element does not always guarantee zero steady-state error. This fact creates prerequisites for design of closed-loop EMS with the desired nonlinear static characteristic and response to the level of the reference input level.

To impart the desired properties, it is possible to combine several nonlinearities within a single EMS. As an example, let us consider the case when the nonlinear activation function is present both in the direct channel of the first-order EMS and in its feedback channel. Such an EMS is described by an equation (4.32) and can be classified as a system with an embedded control algorithm. Equating the right-hand side of the equation to zero allows us to write the following expression for the static characteristic

$$y_1^* - |y_1|^{\alpha_1} \operatorname{sign}(y_1) = 0, \tag{4.646}$$

Moreover, as follows from the analysis of the denominator of the right-hand side of the expression (4.32), when this condition is met in the system, due to the unlimited increase in the gain factor, a sliding mode occurs.

Thus, it can be stated that the nonlinearity at the controller output does not affect the static properties of the system because even in the sliding mode, its output coordinate will be related to the reference signal by the relation (4.43), which in the case under consideration will take the form

$$y_1 = \sqrt[1-\beta_1]{y_1^*} = \sqrt[\alpha_1]{y_1^*}. \qquad (4.647)$$

Summarizing the obtained results and the presented calculations, it can be concluded that the steady-state mode of a system with an irrational activation function coincides with the steady-state mode of a linear or relay system only in specific cases. In other cases, it occupies an intermediate position between the modes in these systems. Moreover, the accuracy of the system is affected by the location of the nonlinear element in the system structure.

4.4. Study of time characteristics of EMS

The steady-state values of the coordinates of dynamic systems calculated in the previous section, together with the initial conditions

$$y_1(0) = y_0 \qquad (4.648)$$

form a set of boundary conditions for solving the dynamic equations (4.26) and (4.30).

Having solved these equations, we determine the time characteristics of dynamic systems with an irrational activation function and the main dependencies that define the parameters of the motion trajectory of nonlinear systems.

We will carry out research for the stepwise

$$y_1^* = 1(t), \qquad (4.649)$$

linearly increasing

$$y_1^* = c_1 t \qquad (4.650)$$

and nonlinearly increasing

$$y_1^* = c_2 t^2 \tag{4.651}$$

reference inputs.

4.4.1. System response to a unit step

Let the dynamics of a closed-loop system be described by the expression

$$py_1 = -k_1 \frac{y_1}{|y_1|^\beta} + k_1 y_1^* \tag{4.652}$$

and the reference input is a unit step function.

Assuming that the output coordinate of the object is positive, we represent the equation (4.50) as follows

$$py_1 = -k_1 \frac{y_1}{y_1^\beta} + k_1 1(t) = -k_1 y_1^\alpha + k_1 1(t). \tag{4.653}$$

Let us introduce a new state variable x, defined by the relations

$$x^{1/\alpha} = y_1, \frac{1}{\alpha} x^{\frac{1-\alpha}{\alpha}}; px = py_1 \tag{4.654}$$

and allows us to represent the equation (4.51) in the following form

$$\frac{1}{\alpha} x^{\frac{1-\alpha}{\alpha}} px = -k_1 \left(x^{1/\alpha}\right)^\alpha + k_1 1(t) = -k_1 x + k_1 1(t) \tag{4.655}$$

or

$$x^{\frac{1-\alpha}{\alpha}} px \quad = -\alpha k_1 x + \alpha k_1 1(t). \tag{4.656}$$

In general, the equation (4.54) cannot be solved analytically. Let us consider specific cases of its solution:

• when $a = 1$ the equation (4.54)is transformed into a linear differential equation

$$px = -k_1 x + k_1 1(t), \tag{4.657}$$

the solution of which is known

$$x(t) = k_1\left(1 - e^{-k_1 t}\right). \tag{4.658}$$

By virtue of expressions (4.52) we have $x = y_1$, hence

$$y(t) = k_1\left(1 - e^{-k_1 t}\right). \tag{4.659}$$

• when $a = 0,5$, the equation (4.54) takes the form

$$xpx \quad = -0,5k_1 x + 0,5k_1 1(t). \tag{4.660}$$

Mathematically, equation (4.58) is an Abel equation of the second kind, which can be solved either parametrically or using non-elementary functions. Such non-elementary functions include the Lambert W function, which is the inverse of

$$f(x) = xe^x, \tag{4.661}$$

i.e. it allows solving an equation of the form

$$xe^x = y \tag{4.662}$$

in the form

$$x = \text{Lambert}\,W(y). \tag{4.663}$$

The solution to the equation (4.58) using the Lambert W function is as follows:

$$x = \text{Lambert}\,W\left(e^{-1-\frac{k_1 t}{2}-\frac{k_1 A_1}{2}}\right)+1, \tag{4.664}$$

where A_1 is a constant of integration to be determined from the initial conditions.

Based on the expression (4.52), the following relationship can be written

$$y(0) = x^2(0) = 0, \tag{4.665}$$

according to which

$$\operatorname{Lambert}W\left(e^{-1-\frac{k_1 A_1}{2}}\right)+1=0 \tag{4.666}$$

and the desired constant is

$$A_1=-\frac{2j\pi}{k_1}. \tag{4.667}$$

Substituting the constant of integration into the expression (4.62) allows us to represent this expression as follows

$$x=\operatorname{Lambert}W\left(e^{-1-\frac{k_1 t}{2}+j\pi}\right)+1. \tag{4.668}$$

Squaring the last expression determines the desired motion of the closed-loop system.

$$y_1=\left(\operatorname{Lambert}W\left(e^{-1-\frac{k_1 t}{2}+j\pi}\right)+1\right)^2. \tag{4.669}$$

Analysis of the obtained expression shows that the presence of a nonlinear activation function in the feedback leads to the formation of trajectories of the system's movement that differ from exponential ones.

Now let us consider a closed-loop system whose trajectory is described by the equation

$$py_1=\left|y_1^*-y_1\right|^{\alpha},\ \alpha<1 \tag{4.670}$$

and the reference action is a unit step function.

Let us assume the output coordinate of the object is positive. Since the activation function is symmetrical relative to the origin, this assumption is not restrictive, since the results obtained will also be symmetrical relative to the origin. Then the equation (4.68) can be represented as follows

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$$py_1 = \left(y_1^* - y_1\right)^{\alpha}. \tag{4.671}$$

Let us introduce a new variable x, defined by the relations

$$x^{1/\alpha} = y_1^* - y_1, \frac{1}{\alpha} x^{\frac{1-\alpha}{\alpha}} px = py_1^* - py_1. \tag{4.672}$$

Taking into account that a stepwise effect is applied to the input of the system, the relations (4.70) will take the form

$$x^{1/\alpha} = y_1^* - y_1, \frac{1}{\alpha} x^{\frac{1-\alpha}{\alpha}} px = -py_1. \tag{4.673}$$

The relations (4.71) allow us to represent the equation (4.69) in the following form

$$-\frac{1}{\alpha} x^{\frac{1-\alpha}{\alpha}} px = -\left(x^{1/\alpha}\right)^{\alpha} = -x \tag{4.674}$$

or

$$px = \alpha x^{\frac{2\alpha-1}{\alpha}}. \tag{4.675}$$

By replacing the Laplace operator p with the differentiation operator $\frac{d}{dt}$, the equation (4.73) can be reduced to a separable differential equation

$$dt = \frac{dx}{-\alpha x^{\frac{2\alpha-1}{\alpha}}}. \tag{4.676}$$

The solution to the equation (4.74) will be

$$x = \frac{1}{\left(\alpha t - t + A_1\right)^{\frac{\alpha}{\alpha-1}}}, \tag{4.677}$$

and the constant of integration A_1 is determined by the initial conditions from the expression

$$x^2(0) = y_1^* - y_1(0) = y_1^*. \tag{4.678}$$

Substituting the expression (4.75) into the dependence (4.76) at $t = 0$ allows us to determine the constant of integration

$$A_1 = y_1^{*-\frac{\alpha-1}{\alpha}}. \tag{4.679}$$

Taking into account the constant A_1, the equation (4.75) takes the form

$$x = \frac{1}{\left(\alpha t - t + y_1^{*-\frac{\alpha-1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}}. \tag{4.680}$$

Substituting the expression (4.78) into the first dependence of (4.71), we get

$$y_1^* - y_1 = \frac{1}{\left(\left(\alpha t - t + y_1^{*-\frac{\alpha-1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{1}{\alpha}}} \tag{4.681}$$

or

$$y_1 = y_1^* - \frac{1}{\left(\alpha t - t + y_1^{*-\frac{\alpha-1}{\alpha}}\right)^{\frac{1}{\alpha-1}}}. \tag{4.682}$$

The expression (4.80) allows us to state that if an irrational activation function is at the input of the integrator, then the transition function of the system will be a power-law dependence.

4.4.2. System response to a ramping signal

We will assume that the closed-loop EMS is described by the differential equation

$$py_1 = y_1^* - |y_1|^{\alpha} \operatorname{sign}(y_1), \alpha < 1 \qquad (4.683)$$

and the reference input is a linearly increasing signal (4.48).

Substituting the reference input (4.48) into the equation (4.81)allows us to write a differential equation of the form

$$\frac{dy_1}{dt} = c_1 t - |y_1|^{\alpha} \operatorname{sign}(y_1), \alpha < 1. \qquad (4.684)$$

Assuming that the output coordinate of the object is positive, we can represent the equation (4.82) as follows

$$\frac{dy_1}{dt} = c_1 t - y_1^{\alpha}, \alpha < 1. \qquad (4.685)$$

Let us introduce a new variable x such that

$$x^{\frac{1}{\alpha}} = y_1, \frac{1}{\alpha} x^{\frac{1-\alpha}{\alpha}} \frac{dx}{dt} = -\frac{dy_1}{dt}. \qquad (4.686)$$

The relations (4.84) allow us to represent the equation (4.83) in the following form

$$-\frac{1}{\alpha} x^{\frac{1-\alpha}{\alpha}} \frac{dx}{dt} = c_1 t - \left(x^{\frac{1}{\alpha}} \right)^{\alpha} = c_1 t - x \qquad (4.687)$$

Direct solution of the equation (4.85) in the general case is difficult, so we will first consider several special cases.

- When $\alpha = 1$ the considered EMS with a nonlinear activation function degenerates into a linear EMS, the motion of which is described by the equation

$$-\frac{dx}{dt} = c_1 t - x; \qquad (4.688)$$

or in normal form

$$\frac{dx}{dt} = x - c_1 t. \tag{4.689}$$

The solution to the equation (4.87) is obvious.

$$x = -c_1 + c_1 t + A_1 e^{-t}, \tag{4.690}$$

where A_1 is the integration constant, which is determined from the equation (4.88) at $t = 0, x = 0$;

$$0 = -c_1 + c_1 \times 0 + A_1 e^{-0} \tag{4.691}$$

or

$$c_1 = A_1. \tag{4.692}$$

Then, due to the expression (4.84), the desired output coordinate of the considered EMS will be

$$y_1 = -c_1 + c_1 t + c_1 e^{-t}. \tag{4.693}$$

• When $\alpha = 0,5$ the motion equation of the considered system takes the form

$$-2x\frac{dx}{dt} = c_1 t - x \tag{4.694}$$

or

$$x\frac{dx}{dt} = \frac{1}{2}x - \frac{c_1 t}{2}. \tag{4.695}$$

Equation (4.93) can be reduced to the Abel equation by introducing notations

$$f_1(t) = -\frac{1}{2};\ f_0(t) = \frac{c_1}{2}t \tag{4.696}$$

and new coordinate

$$\xi = \int f_1(t)dt = -\frac{1}{2}t, \tag{4.697}$$

for which the following statement is true

$$f(\xi)=\frac{f_0(t)}{f_1(t)}, \tag{4.698}$$

Resulting Abel equation will have following form

$$x\frac{dx}{d\xi}=x+f(\xi). \tag{4.699}$$

The last equation can be solved only parametrically by introducing a parameter τ such that

$$t=\mathrm{C}e^{-\frac{1}{2}\ln(\tau^2-\tau+c_1)-\frac{\arctan\frac{2\tau-1}{\sqrt{4c_1-1}}}{\sqrt{4c_1-1}}}$$

then

$$x=\mathrm{C}\tau e^{-\frac{1}{2}\ln(\tau^2-\tau+c_1)-\frac{\arctan\frac{2\tau-1}{\sqrt{4c_1-1}}}{\sqrt{4c_1-1}}} \tag{4.700}$$

In other cases, analytical methods for solving the equation (4.83)are not known.

We will assume that the closed-loop EMS is described by the differential equation

$$py_1=\left|y_1^*-y_1\right|^{\alpha}\operatorname{sign}(y_1^*-y_1),\ \alpha<1 \tag{4.701}$$

and the reference input is a linearly increasing signal (4.48).

Substituting the reference input (4.48) into the equation (4.99)allows us to write a differential equation of the form

$$\frac{dy_1}{dt}=\left|c_1t-y_1\right|^{\alpha}sign(c_1t-y_1),\quad \alpha<1. \tag{4.702}$$

Assuming that the output coordinate of the object is positive, we represent the equation (4.100) as follows

$$\frac{dy_1}{dt}=\left(c_1t-y_1\right)^{\alpha},\ \alpha<1. \tag{4.703}$$

Let us introduce a new variable x

$$x^{\frac{1}{\alpha}} = c_1 t - y_1, \quad \frac{1}{\alpha} x^{\frac{1-\alpha}{\alpha}} \frac{dx}{dt} = c_1 - \frac{dy_1}{dt}. \tag{4.704}$$

The relations (4.102) allow us to represent the equation (4.101) in the form

$$c_1 - \frac{1}{\alpha} x^{\frac{1-\alpha}{\alpha}} \frac{dx}{dt} = \left(x^{\frac{1}{\alpha}} \right)^{\alpha} = x \tag{4.705}$$

Direct solution of the equation (4.103) in the general case is difficult, so let us consider a several of special cases:

• When $\alpha = 1$ the considered EMS with a nonlinear activation function degenerates into a linear EMS, the motion of which is described by the equation

$$c_1 - \frac{dx}{dt} = x \tag{4.706}$$

or in normal form

$$\frac{dx}{dt} = -x + c_1. \tag{4.707}$$

The solution to the equation (4.105) is obvious.

$$x = c_1 \left(1 - e^{-t} \right), \tag{4.708}$$

and thus, from the first equation of the system (4.102), the desired output coordinates of the considered EMS can be determined

$$y_1 = c_1 t - c_1 \left(1 - e^{-t} \right) = c_1 \left(t - 1 + e^{-t} \right). \tag{4.709}$$

• When $\alpha = 0{,}5$ the equation of motion of the considered system takes the form

$$c_1 - 2x \frac{dx}{dt} = x. \tag{4.710}$$

or

$$x \frac{dx}{dt} = -\frac{x}{2} + \frac{c_1}{2}. \tag{4.711}$$

The equation (4.109) has no solution in elementary functions, but can be written using the Lambert W function

$$x = c_1\left(\operatorname{Lambert}W\left(\frac{e^{-1-\frac{t}{c_1}-\frac{A1}{c_1}}}{c_1}\right)+1\right). \tag{4.712}$$

Substituting the found value of the coordinate x into the first equation of the system (4.102) at $\alpha = 0,5$, we determine the desired variable of the state of the EMS

$$y_1 = c_1 t - \left[c_1\left(\operatorname{Lambert}W\left(\frac{e^{-1-\frac{t}{c_1}-\frac{A1}{c_1}}}{c_1}\right)+1\right)\right]^2. \tag{4.713}$$

Considering that at the first moment of time $y_1(0) = 0$, as a result of solving the equation (4.110)

$$c_1^2\left(\operatorname{Lambert}W\left(\frac{e^{-1-\frac{t}{c}-\frac{A1}{c_1}}}{c_1}\right)+1\right)^2 = 0 \tag{4.714}$$

we will determine the unknown integration constant

$$A1 = -c_1 ln(-c_1). \tag{4.715}$$

Finally, taking into account the integration constant (4.113), the expression (4.111) will take the form

$$y_1 = c_1 t - \left[c_1\left(\operatorname{Lambert}W\left(\frac{e^{-1-\frac{t}{c_1}+ln(-c_1)}}{c_1}\right)+1\right)\right]^2 \tag{4.716}$$

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or

$$y_1 = c_1 t - \left[c_1 \left(\text{Lambert}W\left(-e^{-1-\frac{t}{c_1}} \right) + 1 \right) \right]^2 . \tag{4.717}$$

As can be seen from the comparison of the results obtained from the special cases of solving equation (4.103), response of linear and nonlinear electromechanical systems to a linearly increasing signal differ significantly, creating the basis for the construction of high-precision tracking electric drives.

Generalizing the results for solving equation (4.103), we introduce the concept of a “pseudo-characteristic equation”.

To explain this concept, let us consider the closed-loop electromechanical system (4.108). Let us represent the second equation of the system (4.108) in the form

$$-2xdx = (x - c_1)dt \tag{4.718}$$

and divide both sides of the resulting equation by $x - c_1$

$$\frac{2x}{c_1 - x} dx = dt. \tag{4.719}$$

The equation (4.117) is a separable variables equation and its solution is obvious

$$-2x - 2c_1 ln(-c_1 + x) + A_1 = t, \tag{4.720}$$

where A_1 is the integration constant.

Thus, finding an unknown motion trajectory x instead of solving a differential equation (4.108) reduces to solving a nonlinear equation (4.118).

Let us introduce a new variable z, for which the relation is true:

$$ln(-c_1 + x) = z, \tag{4.721}$$

then

$$x - c_1 = e^z \tag{4.722}$$

and

$$x = e^{z} + c_1. \qquad (4.723)$$

Comparison of the equation (4.121) with the corresponding equation for the linear system (4.106) shows the identity of the variable z and the product of the root of the characteristic equation, taken with the opposite sign, multiplied by time, i.e. it can be stated that

$$z \equiv -p_1 t, \qquad (4.724)$$

where p_1 is the root of the characteristic equation of a linear closed-loop EMS.

Substituting the expression (4.121) into the equation (4.108), we get

$$-2zc_1 - 2e^{z} - 2c_1 + A_1 - t = 0. \qquad (4.725)$$

The expression (4.122) because of equality (4.123) will be called "pseudo-characteristic equation of the EMS with an irrational activation function for a linearly increasing reference signal."

Table 4.1 shows pseudo-characteristic equations for various degrees of the irrational activation function.

Table 4.1. Pseudo-characteristic equations for different exponents α

Exponent α	Pseudo-characteristic equation
1/2	$-2zc_1 - 2e^{z} - 2c_1 + A_1 - t = 0$
1/3	$-6zc_1^2 - 3(e^{z})^2 - 12c_1e^{z} - 9c_1^2 + 2A_1 - 2t = 0$
1/4	$-12zc_1^3 - 4(e^{z})^3 - 18c_1(e^{z})^2 - 36e^{z}c_1^2 - 22c_1^3 + 3A_1 - 3t$

The numerical coefficients of a pseudo-characteristic equation of arbitrary degree are determined by the order of this degree, and its roots determine the trajectory of the motion of the fictitious coordinate x, and consequently the output coordinate of the closed-loop EMS, which, in accordance with the expression, (4.102) will be

$$y_1 = c_1 t - x^{1/\alpha} = c_1 t - \left(e^z + c_1\right)^{1/\alpha}. \tag{4.726}$$

It should be noted that among the pseudo-characteristic equations listed in Table 4.1, only the first one can be solved analytically with respect to the variable z using known methods. In this case, such a non-elementary function as the Lambert W function is used to express the solution.

Therefore, it is of interest to study the motion of the EMS in the “new” time z, whose dependence on the "old" time t can be determined from the pseudo-characteristic equation

$$t = \alpha f(z), \tag{4.727}$$

where $f(z)$ are the components of the pseudo-characteristic equation that do not contain the variable t.

In the new phase space, the trajectory of the object's motion will be

$$y_1 = c_1 \alpha f(z) - \left(e^z + c_1\right)^{1/\alpha}. \tag{4.728}$$

Thus, the motion trajectories of systems with pseudo-characteristic equations given in Table 4.1 will be as shown in Table 4.2.

Table 4.2. Trajectories of systems for different exponents α

α	Trajectory of movement
1/2	$-\frac{c_1}{2}(2zc_1 + 2e^z + 2c_1 - A_1) - \left(e^z + c_1\right)^2$
1/3	$-\frac{c_1}{3}(6zc_1^2 + 3(e^z)^2 + 12c_1e^z + 9c_1^2 - 2A_1) - \left(e^z + c_1\right)^3$
1/4	$-\frac{c_1}{4}(12zc_1^3 + 4e^{3z} + 18c_1e^{2z} + 36e^z c_1^2 + 22c_1^3 - 3A_1) - \left(e^z + c_1\right)^4$

Since the pseudo-characteristic equations for integer values of the indicator $\frac{1}{\alpha}$ include only components z and e^z, it can be stated that in the new phase space, the processing of the linearly increasing reference signal is carried out linearly with a delay.

4.4.3. System response to a of complex shaped demand signal

The domain of analytical solution of equations (4.26) and (4.30)is limited to the cases considered previously, where the input to the EMS) is either a step or linearly increasing signal. With other reference inputs, it is not possible to find the solution of these equations directly as a function of time t when $\alpha \neq 1$. However, continuing the approach of changing the independent coordinate proposed in the previous section, it is possible to transform the equations of systems (4.26) and (4.30) with an arbitrarily complex changing input signal to an analog of the EMS, whose motion occurs under the influence of a linearly increasing signal.

Let us prove the correctness of such a statement using the example of a first-order linear EMS with a quadratically varying input.

Let the initial motion equations of the EMS have the form

$$\frac{dy_1}{dt} = c_1 t^2 - y_1. \tag{4.729}$$

The solution to the equation (4.127) will be the dependence

$$y_1 = 2c_1 - 2c_1 t + c_1 t^2 + A_1 e^{-t}, \tag{4.730}$$

where the integration constant A_1 is determined from the initial conditions of the equation (4.128) at $t = 0$. We will assume that $y_1(0) = 0$, then

$$0 = 2c_1 + A_1 \tag{4.731}$$

or

$$A_1 = -2c_1. \tag{4.732}$$

Taking into account the found integration constant A_1, the expression (4.128) will take the form

$$y_1 = 2c_1 - 2c_1 t + c_1 t^2 - 2c_1 e^{-t}. \qquad (4.733)$$

Let us show another method to determine the trajectory of motion (4.131).

Let us move on to a new time τ, which is related to time t by a quadratic dependence

$$\tau = t^2 \qquad (4.734)$$

And

$$t = \sqrt{\tau}. \qquad (4.735)$$

Let us find the differential of the expression (4.133)

$$dt = \frac{d\tau}{2\sqrt{\tau}}. \qquad (4.736)$$

Then, in the new time τ, the derivative $\frac{dy_1}{dt}$ will become

$$\frac{dy_1}{dt} = 2\sqrt{\tau}\frac{dy_1}{d\tau}. \qquad (4.737)$$

Taking into account the derivative (4.135) and the ratio (4.132), the equation (4.127) will take the form

$$2\sqrt{\tau}\frac{dy_1}{d\tau} = c_1\tau - y_1. \qquad (4.738)$$

The solution to the last equation is

$$y_1 = c_1\tau - 2c_1\sqrt{\tau} + 2c_1 + A_1 e^{-\sqrt{\tau}}, \qquad (4.739)$$

where at $\tau = 0, y_1(0) = 0$, the constant A_1 takes on a value that is defined by expression (4.130).

Substituting the value of the constant A_1 and the expression (4.132) into the expression (4.137) leads to the previously obtained dependence (4.131), which allows us to judge the correctness of introducing the "new" fictitious time τ for solving the considered equations with an arbitrary input signal.

Let us generalize the obtained results. Let the considered EMS in operator form be described by the equations

$$py_1 = (y_1^* - y_1)^{\alpha} \tag{4.740}$$

And

$$py_1 = y_1^* - y_1^{\alpha}. \tag{4.741}$$

Let us enter new coordinates

$$x_1^{\frac{1}{\alpha}} = y_1^* - y_1, \frac{1}{\alpha} x_1^{\frac{1-\alpha}{\alpha}} px = py_1^* - py_1 \tag{4.742}$$

and

$$x_2^{\frac{1}{\alpha}} = y_1, \frac{1}{\alpha} x_1^{\frac{1-\alpha}{\alpha}} px = py_1 \tag{4.743}$$

Taking into account the notations (4.140) and (4.141), equations (4.138) and (4.139) will take the form

$$py_1^* - \frac{1}{\alpha} x_1^{\frac{1-\alpha}{\alpha}} px = x \tag{4.744}$$

and

$$\frac{1}{\alpha} x_2^{\frac{1-\alpha}{\alpha}} px = y_1^* - x_2. \tag{4.745}$$

Replacing the Laplace operator p with the differentiation operator $\frac{d}{dt}$, the equations (4.142) and (4.143) can be represented as follows

$$\frac{dy_1^*}{dt} - \frac{1}{\alpha} x_1^{\frac{1-\alpha}{\alpha}} \frac{x}{dt} = x \tag{4.746}$$

and

$$\frac{1}{\alpha}x_2^{\frac{1-\alpha}{\alpha}}\frac{x}{dt}=y_1^*-x_2. \tag{4.747}$$

We will assume that the demand signal of arbitrary shape depends on time

$$y_1^*=f(t). \tag{4.748}$$

Let us introduce a new time τ, related to real time by the dependence

$$\tau=f(t). \tag{4.749}$$

Then the differential will be true

$$d\tau=\frac{df(t)}{dt}dt, \tag{4.750}$$

and derivatives

$$\frac{dt}{d\tau}=\frac{dt}{df(t)}, \quad \frac{dx}{d\tau}=\frac{dx}{dt}\frac{dt}{d\tau} \tag{4.751}$$

From the expression for the derivative $\frac{dx}{d\tau}$, it follows that

$$\frac{dx}{dt}=\frac{dx}{d\tau}\frac{d\tau}{dt}. \tag{4.752}$$

Taking into account the accepted notations, the motion equations of the considered EMS will take the form

$$\tau-\frac{1}{\alpha}x_1^{1/\alpha-1}\frac{dx_1}{d\tau}\frac{d\tau}{dt}=x_1 \tag{4.753}$$

and

$$\frac{1}{\alpha}x_2^{1/\alpha-1}\frac{dx_2}{d\tau}\frac{d\tau}{dt}=\tau-x_2. \tag{4.754}$$

The obtained equations have a form similar to those considered previously. Therefore, for the investigating of EMS with a complex input signals, the following algorithm can be proposed:

• Introduce a new time τ which obeys the dependence (4.146).

• Transform the motion equations from the old-time basis to the new one.

• Use the calculations of the previous section to study systems with a linearly varying input signals.

• Perform the reverse transformation of the obtained results.

4.4.4. Temporal characteristics of high-order EMS

Unlike the trajectories of a closed-loop linear system, the analytical determination of the trajectories of an EMS with a nonlinear activation function by directly solving the corresponding system of differential equations is associated with certain challenges due to the order of the nonlinear system. Therefore, we transition from the equations of motion of the EMS with a nonlinear activation function to the corresponding equations of integral curves. This approach allows us to reduce the order of the considered system and thereby simplify its mathematical description.

Let the motion of an electromechanical object of the n-th order be described by a system of differential equations in Brunovsky form

$$py_1 = y_2;\ py_2 = y_3;\cdots py_{n-1} = y_n;\ py_n = v. \qquad (4.755)$$

By performing the substitution $p \to \frac{d}{dt}$, we bring the equations (4.153) to the form

$$\frac{dy_1}{dt} = y_2;\ \frac{dy_2}{dt} = y_3;\cdots\frac{dy_{n-1}}{dt} = y_n;\ \frac{dy_n}{dt} = v. \qquad (4.756)$$

and expressing the differential of the independent variable dt through the state variables

$$\frac{y_1}{y_2} = dt;\ \frac{y_2}{y_3} = dt;\cdots\frac{y_{n-1}}{y_n} = dt;\ \frac{y_n}{v} = dt, \qquad (4.757)$$

we get

$$\frac{dy_1}{y_2} = \frac{dy_2}{y_3} = \cdots = \frac{dy_{n-1}}{y_n} = \frac{dy_n}{v}. \qquad (4.758)$$

By choosing the variable y_1 as an independent one and using the relations (4.154), we can write a system of equations of the form

$$\frac{dy_2}{dy_1}=\frac{y_3}{y_2};\cdots\frac{dy_{n-1}}{dy_1}=\frac{y_n}{y_2};\frac{dy_n}{dy_1}=\frac{v}{y_2} \tag{4.759}$$

or in general terms

$$\frac{dy_j}{dy_1}=\frac{y_{j+1}}{y_2},\ j=2,\cdots,n-1;$$
$$\frac{dy_n}{dy_1}=\frac{v}{y_2}. \tag{4.760}$$

The resulting system describes the integral curves of the considered EMS. The system of equations (4.158) does not contain a variable t and therefore has a dimension $n-1$. Such a system of equations is convenient to use when controlling the full vector of state variables. In the case when the control input v does not depend on the k-th coordinate, the fraction $\frac{dy_k}{y_{k+1}}$ can be excluded from the relations (4.156). This also reduces the order of the system (4.158) and simplifies its description.

Thus, the order of the system of equations describing the integral curves of the considered EMS is at least one less than the order of the original system of motion equations and allows us to determine the variable states of the system through a new independent coordinate y_1. Subsequent substitution of the found solutions into the equations (4.154) and their integration over time allows us to determine the unknown trajectories of motion.

We will consider the peculiarities of solving the equations describing the motion of the EMS in the phase space formed by the state variables of the electromechanical object using the example of a second-order EMS, the dynamics of which in Brunovsky form is described by equations.

$$py_1 = y_2;$$
$$py_2 = v \tag{4.761}$$

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and the control input in the general case has the form

$$v = \sqrt[\alpha]{y_1^* - k_1 y_1 - k_2 y_2}\, sign(y_1^* - k_1 y_1 - k_2 y_2). \tag{4.762}$$

We will assume that the coefficients k_1 and k_2 are selected to ensure asymptotic behavior of transient processes in the system, i.e. the condition is satisfied along the entire trajectory of motion

$$y_1^* - k_1 y_1 - k_2 y_2 \geq 0. \tag{4.763}$$

This assumption allows us to represent the equations (4.159) together with the algorithm (4.159) as follows

$$\begin{aligned} py_1 &= y_2; \\ py_2 &= \sqrt[\alpha]{y_1^* - k_1 y_1 - k_2 y_2}. \end{aligned} \tag{4.764}$$

In the case where the condition (4.161) is not met, the dynamics of the system (4.162) will be described by the equations

$$\begin{aligned} py_1 &= y_2; \\ py_2 &= \begin{cases} \sqrt[\alpha]{y_1^* - k_1 y_1 - k_2 y_2}, & if \quad y_1^* - k_1 y_1 - k_2 y_2 \geq 0; \\ -\sqrt[\alpha]{-y_1^* + k_1 y_1 + k_2 y_2}, & if \quad y_1^* - k_1 y_1 - k_2 y_2 < 0, \end{cases} \end{aligned} \tag{4.765}$$

i.e. the switching line $y_1^* - k_1 y_1 - k_2 y_2 = 0$ splits the system (4.163) into two subsystems

$$\begin{aligned} py_1 &= y_2; \\ py_2 &= \sqrt[\alpha]{y_1^* - k_1 y_1 - k_2 y_2} \end{aligned} \tag{4.766}$$

and

$$\begin{aligned} py_1 &= y_2; \\ py_2 &= -\sqrt[\alpha]{-y_1^* + k_1 y_1 + k_2 y_2}, \end{aligned} \tag{4.767}$$

which are symmetrical with respect to the switching line. Dividing the second equation of the system (4.162) by the first, we eliminate the independent variable t

$$y_2 \frac{\partial y_2}{\partial y_1} = \sqrt[\alpha]{y_1^* - k_1 y_1 - k_2 y_2}. \tag{4.768}$$

Using known analytical methods does not allow us to determine the solution of the differential equation (4.166). However, solutions can be found for a number of special cases.

Let us consider these solutions.

• Let the feedback coefficients k_1 and k_2 be equal to zero. In this case, the considered EMS is described by a system of differential equations

$$py_1 = y_2; \quad py_2 = \sqrt[\alpha]{y_1^*} \tag{4.769}$$

and corresponds to an open-loop system, the block diagram of which is shown in Fig. 4.9.

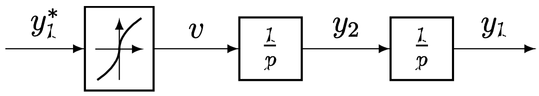

Fig. 4.9. Block diagram of open-loop second-order EMS

The analytical solution of the system of equations (4.167) with respect to y_1 under a constant reference input, obtained without reducing the order, has the form

$$y_1 = \frac{1}{2}\left(y_1^*\right)^{\frac{1}{\alpha}} t^2 + A_1 t + A_2, \tag{4.770}$$

where the integration constants A_1 and A_2 are determined from the initial conditions

$$y_1(0) = 0, \left.\frac{dy_1}{dt}\right|_{t=0} = 0 \tag{4.771}$$

by ratios

$$A_1 = 0, A_2 = 0. \tag{4.772}$$

Thus, the equation (4.168) will take the form

$$y_1 = \frac{1}{2}\sqrt[\alpha]{y_1^*}t^2, \tag{4.773}$$

The expression (4.171) agrees well with the previously obtained results of the study of open-loop systems and is obvious. We will consider the obtained results as reference.

Now we solve the system (4.167) by reducing its order. To do this, dividing the second equation by the first, we get

$$\frac{dy_2}{dy_1} = \frac{\sqrt[\alpha]{y_1^*}}{y_2} \tag{4.774}$$

or

$$y_2 \frac{dy_2}{dy_1} = \sqrt[\alpha]{y_1^*}. \tag{4.775}$$

As already mentioned, equations of the form (4.173) are Abel equations of the second kind and have a solution

$$y_2(y_1) = \pm\sqrt{2\left(y_1^*\right)^{\frac{1}{\alpha}} y_1 + A_1}. \tag{4.776}$$

Considering the initial conditions $A_1 = 0$

$$y_2(y_1) = \pm\sqrt{2\left(y_1^*\right)^{\frac{1}{\alpha}} y_1}. \tag{4.777}$$

Substituting the expression (4.175) into the first equation of the system of equations (4.167)

$$\frac{dy_1}{dt} = y_2(y1) = \pm\sqrt{2\left(y_1^*\right)^{\frac{1}{\alpha}} y_1}, \tag{4.778}$$

we will reduce the resulting equation to a separable variable equation

$$\frac{dy_1}{\sqrt{2y_1^{*1/\alpha}y_1}} = \pm dt, \tag{4.779}$$

the solution of which

$$y_1 = \frac{1}{2}y_1^{*1/\alpha}(t + A_2)^2 \tag{4.780}$$

taking into account the initial conditions, takes a form that fully corresponds to the previously obtained expression

$$y_1 = \frac{1}{2}\sqrt[\alpha]{y_1^*}\,t^2. \tag{4.781}$$

• Let us close the considered EMS by feedback on the output coordinate y_1 (Fig. 4.10).

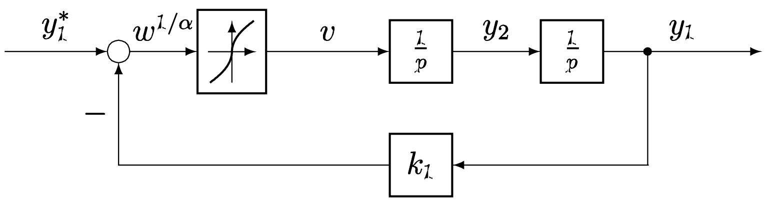

Fig. 4.10. Block diagram of a second-order closed-loop EMS

Mathematically, this corresponds to a nonzero coefficient k_1 in the motion equations (4.162)

$$py_1 = y_2;$$
$$py_2 = \sqrt[\alpha]{y_1^* - k_1 y_1}. \tag{4.782}$$

In the equations (4.180) we will perform the substitution

$$w^\alpha = y_1^* - k_1 y_1;\ \alpha w^{\alpha-1} pw = py_1^* - k_1 py_1 \tag{4.783}$$

and we will represent the system (4.180) as follows

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$$\frac{py_1^* - \alpha w^{\alpha-1} pw}{k_1} = y_2;$$
$$py_2 = w. \qquad (4.784)$$

Assuming that the reference action y_1^* does not change, we represent the system (4.182) in normal form

$$pw = -\frac{k_1 y_2}{\alpha w^{\alpha-1}};$$
$$py_2 = w \qquad (4.785)$$

and write down the equation of the integral curve

$$\frac{dy_2}{dw} = -\frac{\alpha w^{\alpha}}{k_1 y_2}, \qquad (4.786)$$

which is a separable variable equation

$$k_1 y_2 dy_2 = -\alpha w^{\alpha} dw \qquad (4.787)$$

the solution of which is obvious

$$\int k_1 y_2 dy_2 = -\int \alpha w^{\alpha} dw,$$
$$\frac{1}{2} k_1 y_2^2 = -\frac{\alpha}{\alpha+1} w^{\alpha+1},$$
$$y_2 = \pm \frac{\sqrt{2}}{k_1(\alpha+1)} \sqrt{-\alpha(\alpha+1) k_1 w^{\alpha+1}}. \qquad (4.788)$$

Taking into account the notation (4.181), the last expression can be represented as follows

$$y_2 = \pm \frac{\sqrt{2}}{k_1(\alpha+1)} \sqrt{-\alpha(\alpha+1) k_1 (y_1^* - k_1 y_1)^{\frac{\alpha+1}{\alpha}}}, \qquad (4.789)$$

where the "+" sign is taken when the coordinate y_1 increases, and "–" when it decreases.

Substituting the found expression into the first equation of the system (4.180), we get a differential equation for determining the unknown coordinate y_1

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$$py_1 = \pm \frac{\sqrt{2}}{k_1(\alpha+1)} \sqrt{-\alpha(\alpha+1)k_1(y_1^* - k_1 y_1)^{\frac{\alpha+1}{\alpha}}} . \quad (4.790)$$

The solution of the equation (4.188) determines the motion trajectory $y_1(t)$ in the class of exponential functions

$$y_1 = \frac{\left(k_1 t^2 (\alpha+1)^2 \exp(\alpha,t) + 2y_1^*\alpha^2 + 2y_1^*\alpha\right)}{2k_1\alpha(\alpha+1)}, \alpha \neq 1 \quad (4.791)$$

where

$$exp(\alpha,t) = e^{-\frac{ln\left(-\frac{2\alpha(\alpha+1)}{k_1(\alpha-1)^2 t^2}\right)}{(1-1/\alpha)\alpha}}, \quad \alpha \neq 1 \quad (4.792)$$

When $\alpha = 1$, the equation (4.188) takes the form

$$py_1 = \pm \frac{1}{2} \frac{\sqrt{-4k_1 w^2}}{k_1}, \quad (4.793)$$

and its solution, taking into account (4.181), will be

$$y_1 = \frac{e^{\sqrt{-k_1}t} + y_1^*}{k_1}. \quad (4.794)$$

Analysis of the expression (4.190) shows that the range of values of the functions $\exp(\alpha,t)$ and $e^{\sqrt{-k_1}t}$ is the range of complex numbers, which corresponds to the occurrence of oscillations on the curve of the transient process. This fact is beyond doubt, since when $\alpha = 1$ the considered EMS turns into a well-known conservative system with a transfer function

$$W(p) = \frac{K}{Tp^2 + 1}, \quad (4.795)$$

and when $\alpha \neq 1$, it can be classified as a conservative dynamic system of the second order with a variable time constant depending on the control error

$$W(p)=\frac{K}{T\left(y_1^*-y_1\right)p^2+1}. \tag{4.796}$$

• Now let the coefficient k_2 be non-zero, and the coefficient k_1 - to be equal to zero. In this case, the considered EMS has one zero root of the characteristic equation, and its dynamics is described by the following equations

$$py_1=y_2;$$
$$py_2=\sqrt[\alpha]{y_1^*-y_2}. \tag{4.797}$$

The corresponding block diagram is shown in Fig. 4.11.

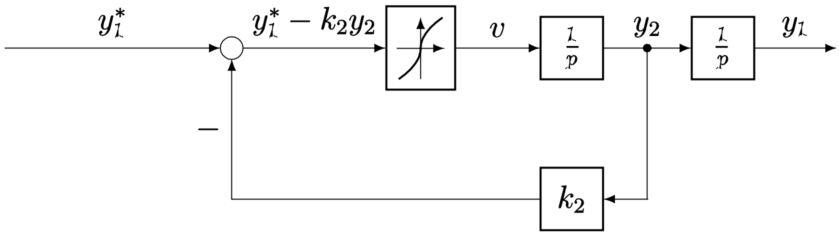

Fig. 4.11. Block diagram of a second-order closed-loop EMS with a zero root of the characteristic equation

Generally speaking, to determine the trajectories of the system shown in Fig. 4.11, it is convenient to use the calculations obtained in the study of first-order dynamic systems. In this case, the coordinate y_2 is determined, and then, by integrating it, the variable y_1 is determined.

However, this approach involves solving two differential equations and contradicts the considered methodology of studying EMS using integral curves. Let us compose an equation of the integral curve for the system (4.195)

$$\frac{dy_2}{dy_1}=\frac{\sqrt[\alpha]{y_1^*-y_2}}{y_2}. \tag{4.798}$$

Let us introduce a notation for the expression under the radical

$$w^\alpha=y_1^*-y_2 \tag{4.799}$$

and write down its differential

$$\alpha w^{\alpha-1}dw = dy_1^* - dy_2. \tag{4.800}$$

With a constant reference input, the expression (4.198) will take the form

$$-\alpha w^{\alpha-1}dw = dy_2. \tag{4.801}$$

Taking into account the differential (4.199), the expression (4.196) will be

$$-\frac{\alpha w^{\alpha-1}dw}{dy_1} = \frac{w}{y_1^* - w^{\alpha}} \tag{4.802}$$

or

$$\frac{dw}{dy_1} = -\frac{w^{2-\alpha}}{\alpha(y_1^* - w^{\alpha})}. \tag{4.803}$$

The equation (4.201) is a separable equation and can be represented as follows

$$dy_1 = -\frac{\alpha(y_1^* - w^{\alpha})}{w^{2-\alpha}}dw. \tag{4.804}$$

Then, the solution to the equation (4.202) will be the dependence

$$y_1 = \frac{\alpha w^{2\alpha-1}}{2\alpha-1} - \frac{\alpha w^{\alpha-1}y_1^*}{\alpha-1}. \tag{4.805}$$

Taking into account the dependence (4.197), the expression (4.203) can be represented as follows

$$y_1 = \frac{\alpha(y_1^* - y_2)^{2-1/\alpha}}{2\alpha-1} - \frac{\alpha(y_1^* - y_2)^{1-1/\alpha}y_1^*}{\alpha-1}. \tag{4.806}$$

The dependence (4.204) found in this way can be used in two ways:

1. To determine the coordinate y_2 in a function y_1 with subsequent solution of the differential equation

$$py_1 = y_2. \tag{4.807}$$

2. To directly determine the desired coordinate y_1, excluding the solution of the differential equation (4.808)

However, in this case the coordinate y_2 must be a known function of time, i.e.

determined according to the calculations presented earlier.

Analysis of the equation (4.205) shows that the first method is reduced to solving a nonlinear equation (4.204) with respect to y_2, which in general has no analytical solution. Therefore, it is preferable to use the second method together with the calculations presented in the previous section.

The considered expressions and relations show that in the general case, for arbitrary values of the exponent α, it is not possible to determine an analytical expression for the transition function of a system with internal feedback and an irrational activation function. Therefore, further research will include the results of numerical calculations.

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CHAPTER 5
SYNTHESIS OF CLOSED-LOOP ELECTROMECHANICAL SYSTEMS WITH NON-LINEAR ACTIVATION FUNCTION

5.1. Motion of characteristic equation roots of a closed-loop system with a non-linear activation function

5.1.1. The concept of free root movement

The use of the root method for the synthesis of control systems is, on the one hand, fairly simple and informative, and on the other hand, a powerful tool for setting the desired dynamics of a closed-loop system. Moreover, currently, only time-invariant characteristic equations are used in modal synthesis, the coefficients of which do not change during the operation of the system. However, there are studies devoted to dynamic systems with free movement of the roots of the characteristic equation [10], i.e. systems in which the values of the coefficients, and therefore the roots of the characteristic equation, constantly change as a function of state variables, control inputs, time, etc. Such characteristic equations can be described by a dependence of the form

$$\sum_{i=0}^{n} a_i(\eta, u, t, \cdots) p_i = 0. \tag{5.809}$$

From the control theory perspective, equations whose coefficients change as a function of state variables are of the greatest interest

$$\sum_{i=0}^{n} a_i(\eta) p_i = 0. \tag{5.810}$$

Let us consider the change in the coefficients of the characteristic equation of a system with an irrational activation function using the example of a first-order dynamic object

$$py_1 = a_{11} y_1 + m_1 U. \tag{5.811}$$

Let's assume that control is being applied to the object

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$$U = -k\left|\eta_1\right|^{\alpha} \operatorname{sign}\left(\eta_1\right), \tag{5.812}$$

where $\eta_1 = y_1 - y_1^*$.

Non-linear control (5.4) can be represented in a form similar to linear control if we introduce the concept of "variable gain factor of the controller"

$$U = -g(\eta_1)\eta_1. \tag{5.813}$$

Dividing the control input (5.4) by the control input (5.5), we obtain

$$1 = \frac{-k\left|\eta_1\right|^{\alpha} \operatorname{sign}\left(\eta_1\right)}{-g(\eta_1)\eta_1} \tag{5.814}$$

where

$$g\left(\eta_1\right) = \frac{k \cdot \left|\eta_1\right|^{\alpha}}{\left|\eta_1\right|} = \frac{k}{\left|\eta_1\right|^{1-\alpha}}. \tag{5.815}$$

Substituting the control algorithm (5.5) into the equation (5.3), we get a system of equations that describes the dynamics of a closed-loop electromechanical system

$$\begin{aligned} &py_1 = a_{11}y_1 + m_1U; \\ &U = -g\left(\eta_1\right)\eta_1. \end{aligned} \tag{5.816}$$

The block diagram of the closed-loop system, which is described by the equations (5.8), is shown in Fig. 5.1. The corresponding transfer function can be defined as follows

$$\Phi\left(p\right) = \frac{g\left(\eta_1\right)m_1 \dfrac{1/p}{1 - a_{11}/p}}{1 + g\left(\eta_1\right)m_1 \dfrac{1/p}{1 - a_{11}/p}} = \frac{\dfrac{g(\eta_1)m_1}{g(\eta_1)m_1 - a_{11}}}{\dfrac{1}{g(\eta_1)m_1 - a_{11}}p + 1}. \tag{5.817}$$

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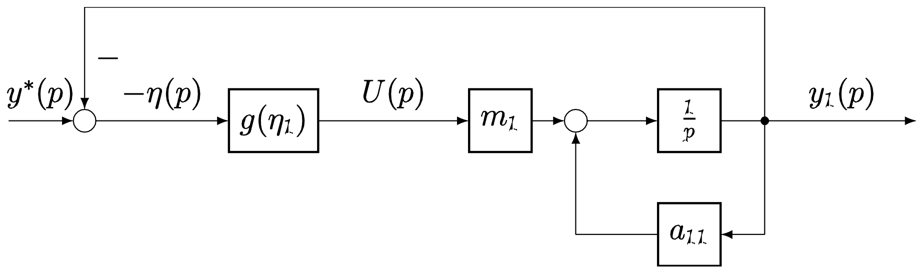

Fig.5.1. Block diagram of the investigated control system

Comparison of the transfer function (5.9) with the transfer function of the aperiodic element

$$W(p) = \frac{K}{Tp+1} \qquad (5.818)$$

shows that the investigated control system is equivalent to a first-order aperiodic element with a gain factor and a time constant

$$K = \frac{g(\eta_1)m_1}{g(\eta_1)m_1 - a_{11}}; \quad T = \frac{1}{g(\eta_1)m_1 - a_{11}}. \qquad (5.819)$$

Taking into account the dependence (5.7), the parameters (5.11)can be represented as follows

$$K(\eta_1) = \frac{m_1 k|\eta_1|^{\alpha}}{m_1 k|\eta_1|^{\alpha} - a_{11}|\eta_1|} = \frac{m_1 k}{m_1 k - a_{11}|\eta_1|^{1-\alpha}};$$

$$T(\eta_1) = \frac{|\eta_1|}{m_1 k|\eta_1|^{\alpha} - a_{11}|\eta_1|} = \frac{1}{m_1 k|\eta_1|^{\alpha-1} - a_{11}} = \frac{|\eta_1|^{1-\alpha}}{m_1 k - a_{11}|\eta_1|^{1-\alpha}}.$$

(5.820)

The expressions (5.12) show the change in the parameters of a closed-loop system as it moves from any initial position to the steady-state point, which is determined by solving the equation of steady motion of a closed-loop system

$$a_{11}y_1 + m_1 g(\eta_1)\eta_1 = 0. \qquad (5.821)$$

When $\alpha = 1$, expressions (5.12) are converted into the well-known dependencies

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$$K = \frac{m_1 k}{m_1 k - a_{11}}; \quad T(\eta_1) = \frac{1}{m_1 k - a_{11}} \tag{5.822}$$

in order to determine the gain and time constant of a linear control system with an algorithm

$$U = -(k\eta_1). \tag{5.823}$$

Thus, the parameters of a control system with an irrational activation function, which are described by the expressions (5.12), are a generalization of the known results obtained for linear electromechanical systems.

Let us consider the values of the parameters (5.12) when the system moves from a point in phase space lying on the boundary of the system's domain of definition to the origin. At the first moment of time, the deviation η_1 becomes equal to one

$$\eta_1 = y_1\Big|_{y_1 = 1} - y_1^*\Big|_{y_1^* = 0} = 1, \tag{5.824}$$

and the expressions $K(\eta_1)$ and $T(\eta_1)$ are respectively equal to

$$K(\eta_1)\Big|_{\eta_1 = 1} = \frac{m_1 k}{m_1 k - a_{11}}; \quad T(\eta_1)\Big|_{\eta_1 = 1} = \frac{1}{m_1 k - a_{11}}. \tag{5.825}$$

Comparison of expressions (5.14) and (5.17) shows their identity and allows us to conclude that the movement of a system with an irrational activation function begins at the boundary of the domain with parameters corresponding to the parameters of a linear system.

If the control system reaches the origin, then the deviation η_1 becomes zero, and the values of parameter (5.12) can be determined by considering the limits

$$\lim_{\eta_1 \to 0} K(\eta_1) = \lim_{\eta_1 \to 0} \frac{m_1 k}{m_1 k - a_{11}|\eta_1|^{1-\alpha}} = \frac{m_1 k}{m_1 k} = 1;$$

$$\lim_{\eta_1 \to 0} T(\eta_1) = \lim_{\eta_1 \to 0} \frac{|\eta_1|^{1-\alpha}}{m_1 k - a_{11}|\eta_1|^{1-\alpha}} = 0. \tag{5.826}$$

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Analysis of the expressions (5.18) shows that when the origin is reached, the system (5.8) becomes inertialess with a unit gain factor, i.e., it is equivalent to a relay system operating in a sliding mode. However, taking into account that the control system (5.8) with the algorithm (5.4) is static, achieving a zero steady-state error in it is possible only for very small values of the exponent α. In other cases, for the considered closed-loop system, the domain of definition of the functions (5.12) is limited, on the one hand, by the initial deviation of the system $\eta_1(0)$, and on the other – by the point of the steady-state mode $\eta_1(\infty)$.

An analysis of the results of mathematical modeling of the studied control system with single parameters under a single reference input allows us to construct the dependences of the equivalent gain factor and time constant of the system on the control error, shown in Fig. 5.2.

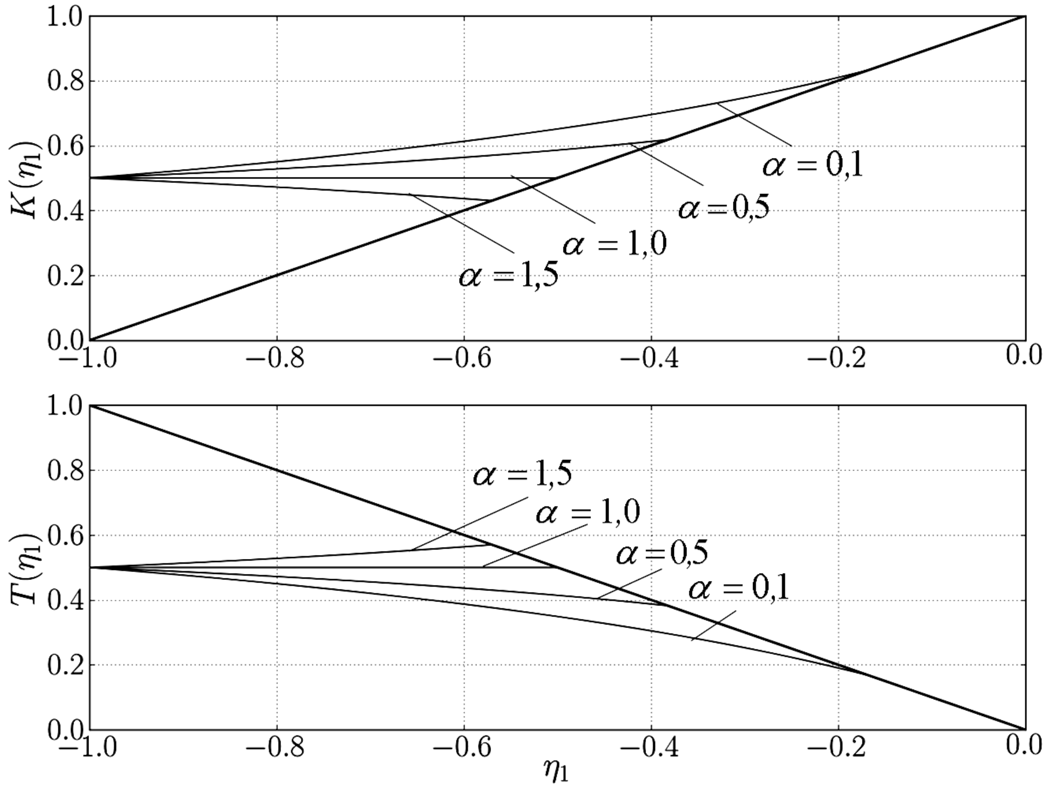

Fig. 5.2. Dependences of equivalent dynamic parameters $K(\eta_1)$ and $T(\eta_1)$ on the deviation η_1

Often, when analyzing the dynamic characteristics of control systems, instead of analyzing the time constants, the roots of the characteristic equation are considered. For the considered closed-loop system, the characteristic equation can be represented as follows

$$\frac{p}{g(\eta_1)m_1 - a_{11}} + 1 = 0. \tag{5.827}$$

The solution of the equation (5.19) allows us to determine the desired value of the root

$$p(\eta_1) = -g(\eta_1)m_1 + a_{11} \tag{5.828}$$

or taking into account the expression (5.7)

$$p(\eta_1) = -\frac{m_1 k - a_{11}|\eta_1|^{1-\alpha}}{|\eta_1|^{1-\alpha}}. \tag{5.829}$$

The graph of the dependence of the root of the characteristic equation (5.21) on the deviation η_1 is shown in Fig. 5.3.

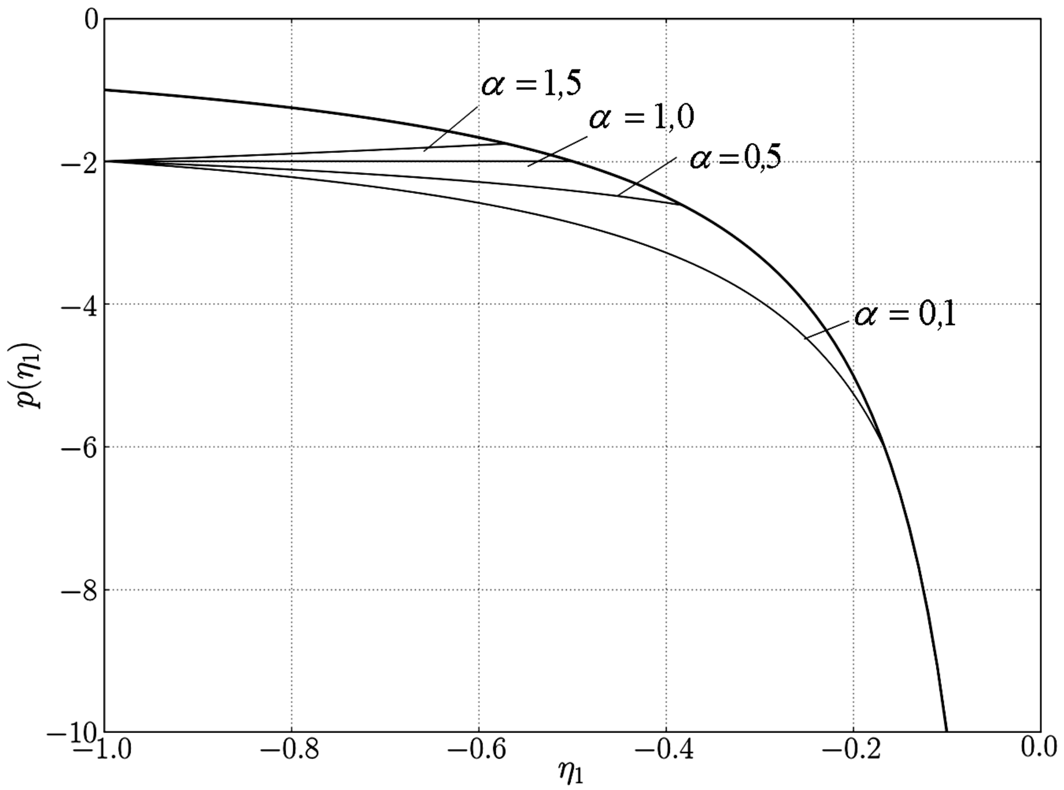

Fig. 5.3. Movement of the root of the characteristic equation

The analysis of the dependencies shown in Fig. 5.2 and 5.3 confirms the previously made conclusions about the reduction of the static error of the system for the control input with a decrease in the exponent α and allows us to formulate several theorems on the equivalent dynamic parameters of closed-loop systems with irrational activation functions.

Theorem 1. *The values of the gain factor of a closed-loop system* $K(\eta_1)$ *in steady-state mode with a constant reference input* $y_1^*(t)$ *and different exponents* α *lie on a single straight line.*

Proof. To prove the theorem, we define the gain of a closed-loop system as the ratio of the output signal to the input signal in steady-state mode.

$$K_e = \left.\frac{y_1(t)}{y_1^*(t)}\right|_{t\to\infty}. \tag{5.830}$$

Taking into account the expression (5.3), the coefficient (5.22) can be represented as follows

$$K_e = \left.\frac{y_1^*(t) + \eta_1(t)}{y_1^*(t)}\right|_{t\to\infty} \tag{5.831}$$

or

$$K_e = 1 + \left.\frac{\eta_1(t)}{y_1^*(t)}\right|_{t\to\infty}. \tag{5.832}$$

The linear dependence of the gain K on the deviation η_1 at a constant reference signal $y_1^*(t)$ in steady-state mode has been proven.

Theorem 2. *The values of the time constant of a closed-loop system* $T(\eta_1)$ *in steady-state with a constant reference signal* $y_1^*(t)$ *and different exponents* α *lie on a single straight line.*

Proof. Let us write the transfer function of the closed-loop control system for a generalized first-order dynamic object

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$$\Phi(p) = \frac{gK_1/T_1 p+1}{1+gK_1/T_1 p+1} = \frac{gK_1}{T_1 p+1+gK_1} =$$
$$= \frac{gK_1/gK_1+1}{1+pT_1/gK_1+1} \tag{5.833}$$

and write down the equivalent gain and time coefficients

$$K_e = \frac{gK_1}{gK_1+1};$$
$$T_e = T_1/gK_1+1. \tag{5.834}$$

Having equated the values of the coefficients (5.23) and (5.26)

$$\frac{gK_1}{gK_1+1} = \left.\frac{y_1^*(t)+\eta_1(t)}{y_1^*(t)}\right|_{t\to\infty} \tag{5.835}$$

Let us find the component gK_1+1

$$gK_1+1 = -\left.\frac{y_1^*(t)}{\eta(t)}\right|_{t\to\infty}. \tag{5.836}$$

The expression (5.28) allows us to represent the equivalent time constant T_e as follows

$$T_e = -T_1\left.\frac{\eta(t)}{y_1^*(t)}\right|_{t\to\infty}. \tag{5.837}$$

The linear dependence of the equivalent time constant on the value of the steady-state error of a closed-loop system is proven for a constant reference signal $y_1^*(t)$.

Taking into account that for the considered control object the root of the characteristic equation of the closed-loop system is determined by an expression of the form

$$p_e = -\frac{1}{T_e} \tag{5.838}$$

we can formulate a corollary to Theorem 2: *The values of the root of the* *characteristic equation of a closed-loop system* $p(\eta_1)$ *in steady-state with a constant reference signal* $y_1^*(t)$ *and different exponents* α *lie on a single hyperbola.*

The proof of this statement is obvious and follows from the substitution of the time constant (5.29) into the expression (5.30).

The analysis of the results obtained when proving the theorems formulated above shows that the parameters of a control system with an irrational activation function depend on two quantities – the deviation $\eta_1(t)$ and the reference input $y_1^*(t)$. It should be noted that these quantities are not independent. The deviation value $\eta_1(t)$ is determined by the parameters of the control object, the reference input and the exponent α. In general case, the deviation $\eta_1(t)$ is also affected by the disturbance. Therefore, the expressions (5.24), (5.29) and (5.30) can be represented by the following dependencies

$$K_e = 1 + f_K\left(y_1^*(t), \alpha\right)\Big|_{t \to \infty};$$

$$T_e = -T_1 f_T\left(y_1^*(t), \alpha\right)\Big|_{t \to \infty};$$

$$p_e = f_p\left(y_1^*(t), \alpha\right)\Big|_{t \to \infty}. \qquad (5.839)$$

The functions $f_K(.),\ f_T(.),\ f_p(.)$ are determined as a result of substitution the results of solving the equation of static equilibrium of a closed-loop system (5.8) into expressions (5.24), (5.29) and (5.30).

Summarizing the above calculations, the following conclusions can be made: the steady-state values of the gain and time constant of a closed-loop time-invariant electromechanical system with a non-linear activation function linearly depend on the control error. In transient modes, these dependencies are described by irrational functions (5.12).

5.1.2. Trajectories of free root movement

The calculations given in the previous section show that the roots of the characteristic equation of a system with an irrational activation function change their value depending on the control error. However, in addition to the absolute values of the roots of the characteristic equation, their distribution on the complex plane is important because the same roots, when placed in different quadrants of the complex plane, significantly affect the curve of the transient process, altering it from asymptotically stable to unstable.

Let us consider the influence of the activation function on the distribution of the roots of the characteristic equation of a closed-loop second-order system, the perturbed motion of which is described by the equations

$$p\eta_1 = a_{12}\eta_2; \quad p\eta_2 = a_{21}\eta_1 + a_{22}\eta_2 + m_2 U. \qquad (5.840)$$

For a dynamic object (5.32), optimal control

$$u = f(S) = -k|S|^{\alpha}\,\mathrm{sign}(S) \qquad (5.841)$$

will be searched from the condition of minimization of the integral functional

$$I = \int_0^{\infty}\left(|S|^{1+\alpha} + C|u|^{\frac{1+\alpha}{\alpha}}\right)dt. \qquad (5.842)$$

In the control input (5.33) and the functional (5.34), the switching line is determined by the expression

$$S = \eta_1 + v_{12}\eta_2, \qquad (5.843)$$

where the weighting coefficient v_{12} is related to the coefficients of the Lyapunov function

$$V = \sum_{i,j=1}^{2} V_{ij}\eta_i\eta_j,\ V_{ij} = V_{ji} \qquad (5.844)$$

by the dependency [15]

$$v_{12} = V_{22}/V_{12}. \qquad (5.845)$$

The coefficients of the Lyapunov function V_{in} in accordance with the modified symmetry principle are determined through the parameters of the control object

$$V_{12}=-a_{22}; V_{22}=a_{12}. \tag{5.846}$$

The system of equations (5.32) together with the control input (5.33), equation (5.35) and coefficients (5.37) and (5.38) allows us to write down equations describing the dynamics of a closed-loop electromechanical system with an irrational activation function

$$\begin{aligned} &p\eta_1=a_{12}\eta_2; \\ &p\eta_2=a_{21}\eta_1+a_{22}\eta_2-m_2k\left|\eta_1-\frac{a_{12}}{a_{22}}\eta_2\right|^{\alpha}\operatorname{sign}\left(\eta_1-\frac{a_{12}}{a_{22}}\eta_2\right). \end{aligned} \tag{5.847}$$

Using a variable coefficient

$$g(S)=\frac{|S|^{\alpha}}{S}\operatorname{sign}(S)=|S|^{\alpha-1} \tag{5.848}$$

allows us to represent the system (5.39) as follows

$$\begin{aligned} &p\eta_1 = a_{12}\eta_2; \\ &p\eta_2 = \left[a_{21}-m_2kg(S)\right]\eta_1+\left[a_{22}+\frac{m_2a_{12}}{a_{22}}kg(S)\right]\eta_2. \end{aligned} \tag{5.849}$$

Taking into account the notation (5.35), the coefficient (5.40) will be

$$g(S)=|S|^{\alpha-1}=\left|\eta_1-\frac{a_{12}}{a_{22}}\eta_2\right|^{\alpha-1}. \tag{5.850}$$

Then, denoting the coefficients of the closed-loop system as follows

$$b_{12}=a_{12};b_{21}=a_{21}-m_2kg(S); b_{22}=a_{22}+\frac{m_2a_{12}}{a_{22}}kg(S), \tag{5.851}$$

we will write the system (5.39) in the form

$$p\eta_1=b_{12}\eta_2; \quad p\eta_2=b_{21}\eta_1+b_{22}\eta_2. \tag{5.852}$$

The performed transformations allow us to represent the dynamics of the original closed-loop non-linear system (5.39) in the form of equations (5.44) that are similar to the equations describing the free motion of a linear system.

For the system (5.44) we will create a determinant

$$\Delta(p)=\begin{vmatrix} 0-p & b_{12} \\ b_{21} & b_{22}-p \end{vmatrix}, \tag{5.853}$$

which, when expanded, gives the characteristic polynomial

$$\Delta(p)=p^2-b_{22}p-b_{12}b_{21}. \tag{5.854}$$

The roots of the polynomial (5.46) are determined by the dependencies

$$p_{1,2}=\frac{b_{22}\pm\sqrt{b_{22}^2+4b_{21}b_{12}}}{2}, \tag{5.855}$$

which, taking into account the notations (5.43), will take the form

$$p_{1,2}=\frac{a_{22}}{2}+\frac{m_2 g(S)a_{12}k}{2a_{22}}\pm$$
$$\pm\frac{1}{2}\sqrt{\left(a_{22}+m_2 g(S)k\frac{a_{12}}{a_{22}}\right)^2+4\left(a_{21}-m_2 g(S)k\right)a_{12}}. \tag{5.856}$$

Since the coefficient a_{22} is negative, the first two terms of the expression (5.48) are also negative. Comparing the sum of the first two terms with the radical expression shows that the roots of (5.48) will be negative real numbers if the inequality is satisfied

$$\left(a_{22}+m_2 g(S)k\frac{a_{12}}{a_{22}}\right)^2+4\left(a_{21}-m_2 g(S)k\right)a_{12}\geq 0 \tag{5.857}$$

The expression (5.48) and inequality (5.49) show that the roots of the characteristic equation of a control system with an irrational activation function for different values of the coefficient $g(S)$ can take real and complex values.

The inequality (5.49) allows us, with known parameters of the control object and the controller, by solving the equation

$$m_2^2 a_{12}^2 g(S)^2 k^2/a_{22}^2+\left(2m_2 a_{12}k-4mka_{12}\right)g+a_{22}^2+4a_{12}a_{21}=0,$$

(5.858)

which is obtained from the inequality (5.49), to determine the value of the gain factor $g(S)_b$ that transforms the complex conjugate roots into the real ones

$$g(S)_b = \frac{a_{22}\left(a_{22} \pm 2\sqrt{-a_{12}a_{21}}\right)}{m_2 a_{12} k}. \quad (5.859)$$

Thus, during system startup, the roots of its characteristic equation will be complex conjugate if the equivalent gain of the controller

$$K_e = k \cdot g(S) \quad (5.860)$$

is less than the value

$$K_{eb} = \frac{a_{22}\left(a_{22} \pm 2\sqrt{-a_{12}a_{21}}\right)}{m_2 a_{12}}. \quad (5.861)$$

In this case, the damping properties of the closed-loop electromechanical system decrease and its boost occurs. When the coefficient $g(S)$ reaches the value $g(S)_b$, the roots become real and further movement occurs along asymptotically stable trajectories.

In the case where the movement must initially be carried out along asymptotic trajectories, based on the known initial value of the coefficient $g(S)_0$, the minimum permissible value of the gain factor k_b of the controller can be determined using dependencies similar to (5.51)

$$k_b = \frac{a_{22}\left(a_{22} \pm 2\sqrt{-a_{12}a_{21}}\right)}{m_2 a_{12} g(S)_0}. \quad (5.862)$$

Physically realizable values of the gain factor k_b are those calculated in accordance with the dependence

$$k_b = \frac{a_{22}\left(a_{22} - 2\sqrt{-a_{12}a_{21}}\right)}{m_2 a_{12} g(S)_0}. \quad (5.863)$$

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The above calculations allow us to make several conclusions:

• The use of a variable coefficient $g(S)$, which is a function of the coordinate of the representative point on the equilibrium state line of the controller, allows us to describe the dynamics of a closed-loop electromechanical system with an irrational activation function in the form of differential equations of the free motion of the system. The resulting system of differential equations has a form similar to a system of linear differential equations and can be studied by known methods of the linear theory of dynamic systems.

• The roots of the characteristic equation of a closed-loop electromechanical system with an irrational activation function (CLES IAF) depend not only on the parameters, but also on the coordinates of the perturbed motion of this system. This property of CLES IAF creates the basis for the formation of such an activation function, which, when the system moves, will ensure the desired distribution of its roots and thereby form the required transient process.

• When the values of the controller gain factor are less than that determined in accordance with the dependence (5.54) and for large deviations of the controlled variable, the roots of the characteristic equation of the CLES IAF can be complex conjugate. In this case, the system's stability margin decreases while its response speed increases. This conclusion allows us to formulate the following recommendation: the startup of any dynamic system with an irrational activation function should be carried out in such a way that for large deviations its roots are complex conjugate, and when approaching a given value of the controlled coordinate they become negative real.

• The equivalent gain factor of the controller, at which the transition from the complex-conjugate roots of the characteristic equation of the CLES IAF to real ones occurs, does not depend on the coordinates of the perturbed motion of the system and is determined by its parameters. This conclusion allows us to assert that the change in the type of roots will occur at any initial deviations in the system, thereby allowing the prediction of its motion.

5.1.3. Stability analysis of closed-loop EMS with non-linear activation function

The calculations given in the previous sections show that the roots of the characteristic equation of a non-linear system move along certain trajectories. Moreover, the sliding mode occurs in a system with a large gain factor in in the vicinity of the origin of the phase space. It is obvious that the ways of moving from any initial point of the phase space to the origin of its coordinates are infinitely many, as is the number of trajectories of the roots of the characteristic equation. Taking into account that both elementary and non-elementary functions can be used to specify the variable coefficients of the desired characteristic polynomial, determined by combinations of the trajectories of the roots, the problem of analyzing and synthesizing a closed-loop EMS with a non-linear activation function in general is unsolvable. Therefore, before choosing the reference trajectory of the roots, the type of the desired control input should be specified.

In general, the algorithm of operation of a controller with a non-linear odd activation function is a combination of the state variables of the EMS and the control inputs

$$U = f(y_1^*, y_2^*, \cdots, y_m^*, y_1, y_2, \cdots, y_n), \tag{5.864}$$

moreover, the number of control inputs m is not necessarily equal to the number of state variables n; moreover, in most practical cases, single control input is fed to the controller input y_j^*, which determines the desired level of the controlled variable y_j

$$U = f(y_j^*, y_1, y_2, \cdots, y_n). \tag{5.865}$$

This algorithm allows achieving high dynamic characteristics of EMS and ensures the desired quality of control processes. However, the definition of a function of several variables $f(.)$ is complicated by the lack of an analytical connection between the trajectory of the non-linear system and the roots of its characteristic

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equation.

Therefore, we will consider the control algorithm, which is formed according to the expression

$$U = f(S), \tag{5.866}$$

where the switching line S is determined by a linear combination of the components of the state vector

$$S = a_j y_j^* - \sum_{i=1}^{n} a_i y_i, \ \ S < 1 \ . \tag{5.867}$$

The process of EMS control is characterized by the symmetry of the static characteristic of the controller implementing the algorithm (5.58) relative to the origin. This fact allows us to represent the control (5.58) in the form

$$U = g(|S|)S, \tag{5.868}$$

Where $g(|S|)$ is a variable coefficient

$$g(|S|) = \frac{f(|S|)}{|S|}. \tag{5.869}$$

Substitution of the control input (5.60) into the equations of motion of a generalized electromechanical object of the n-th order, specified in Brunovsky form, allows the equations of the closed-loop EMS to be written as

$$py_1 = y_2; \quad \cdots \quad py_{n-1} = y_n; \quad py_n = g(|S|)S \tag{5.870}$$

or taking into account the expression (5.59)

$$\begin{aligned} &py_1 = y_2; \\ &\vdots \\ &py_{n-1} = y_n; \\ &py_n = g(|S|)\left(a_j y_j^* - \sum_{i=1}^{n} a_i y_i \right). \end{aligned} \tag{5.871}$$

Expanding the brackets in the last equation of the system (5.63), we get

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$$
\begin{aligned}
& py_1 = y_2; \\
& \vdots \\
& py_{n-1} = y_n; \\
& py_n = g(|S|)a_j y_j^* - \sum_{i=1}^{n} a_i g(|S|) y_i.
\end{aligned}
\qquad (5.872)
$$

or taking into account the first $n-1$ equations of the system(5.63)

$$
\begin{aligned}
& py_1 = y_2; \\
& \vdots \\
& py_{n-1} = y_n; \\
& py_n = g(|S|)a_j y_j^* - \sum_{i=1}^{n} a_i g(|S|) p^{i-1} y_1.
\end{aligned}
\qquad (5.873)
$$

From the last equation of the system, it is easy to obtain the characteristic equation of a closed-loop EMS with a non-linear activation function

$$
p^n + \sum_{i=1}^{n} a_i g(|S|) p^{i-1} = 0 \quad . \qquad (5.874)
$$

The coefficients of the characteristic polynomial (5.66) of a closed-loop system are determined by the trajectory of the change in the gain of the non-linear element and the parameters of the switching line that determine this change. Consequently, these quantities also determine the properties of the closed-loop system.

Using the found characteristic polynomial, we will study the stability of the system. For this, we will compose and analyze the Hurwitz determinant and its diagonal minors of a closed-loop system of the third order. In accordance with the expression (5.66), the characteristic equation of such a system will be

$$
p^3 + a_3 g(|S|) p^2 + a_2 g(|S|) p + a_1 g(|S|) = 0. \qquad (5.875)
$$

The Hurwitz determinant for this system is

$$\Delta_3 = \begin{vmatrix} a_3 g(|S|) & a_1 g(|S|) & 0 \\ 1 & a_2 g(|S|) & 0 \\ 0 & a_3 g(|S|) & a_1 g(|S|) \end{vmatrix}, \quad (5.876)$$

and its minors are

$$\Delta_2 = \begin{vmatrix} a_3 g(|S|) & a_1 g(|S|) \\ 1 & a_2 g(|S|) \end{vmatrix} \quad (5.877)$$

and

$$\Delta_1 = a_3 g(|S|). \quad (5.878)$$

We will assume that the coefficients a_i, determining the switching line and the gain of the controller $g(|S|)$ are positive.

Then the positivity of the minor Δ_1 will be obvious.

The minor Δ_2 is positive if the condition is met

$$a_3 a_2 g^2(|S|) - a_1 g(|S|) > 0 \quad (5.879)$$

or

$$a_3 a_2 g(|S|) > a_1. \quad (5.880)$$

The Hurwitz determinant itself satisfies the condition

$$a_3 a_2 a_1 g^3(|S|) - a_1^2 g^2(|S|) > 0, \quad (5.881)$$

which, after simplification, degenerates into the condition (5.72)

$$a_3 a_2 g(|S|) - a_1 > 0. \quad (5.882)$$

As follows from the analysis of the obtained results, as the gain coefficient $g(|S|)$ increases, the value of the Hurwitz determinant and its minors increases, and, as a consequence, the stability margin of the closed-loop EMS also increases. This result is paradoxical from the point of view of classical automatic control theory, but can be explained by an increase in the stability of the system operating in sliding modes, and is confirmed by the results of mathematical modeling of a third order system, which is described by equations (5.65) with unit feedback coefficients a_i and different values of the gain factor $g(|S|)$ (Fig. 5.4–5.7 and 5.8–5.11). Figures 5.4–5.7 reflect the results of studying linear control systems. Fig. 5.8 and 5.9 are provided for a relay control system, and Fig. 5.10 and 5.11 – for a control system with a non-linear activation function.

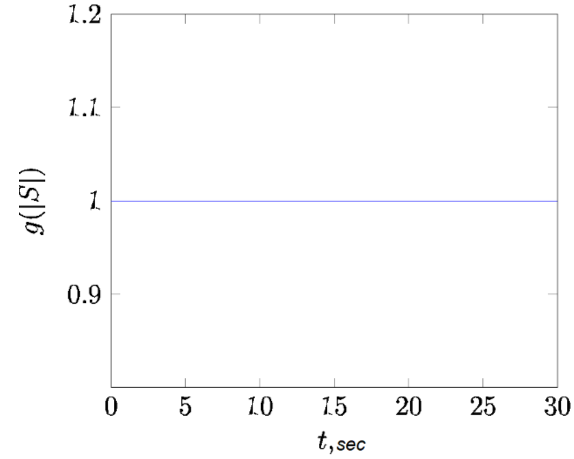

Fig. 5.4. Gain factor $g(|S|)=1$

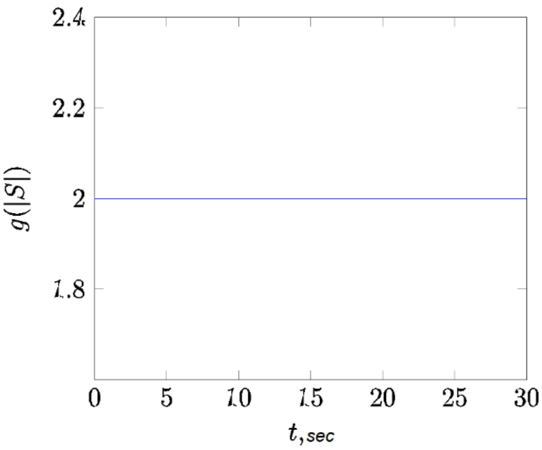

Fig. 5.5. Gain factor $g(|S|)=2$

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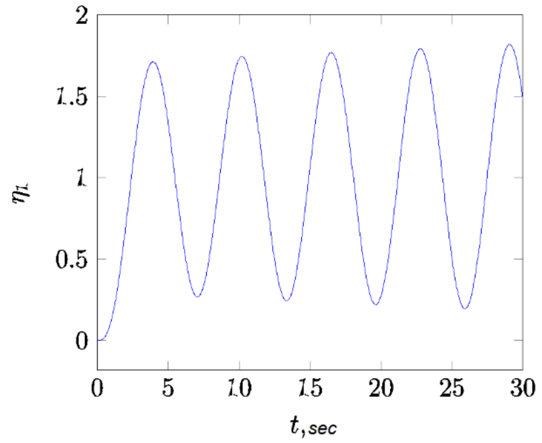

Fig. 5.6. Transient processes with gain factor $g(|S|)=1$

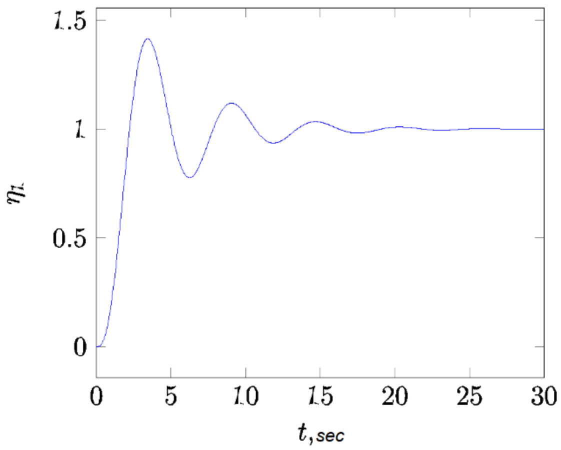

Fig. 5.7. Transient processes with gain factor $g(|S|)=2$

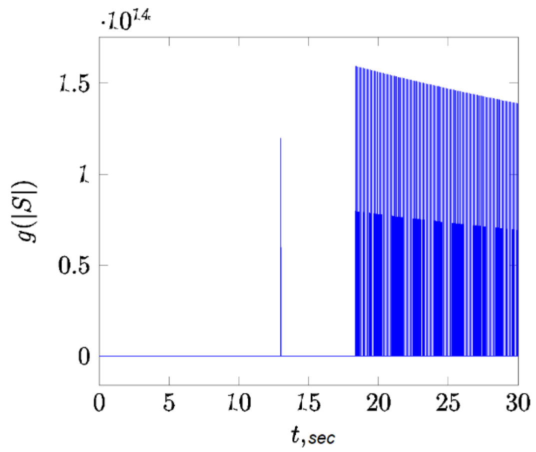

Fig. 5.8. Gain factor $g(|S|)=\dfrac{1}{|S|}$

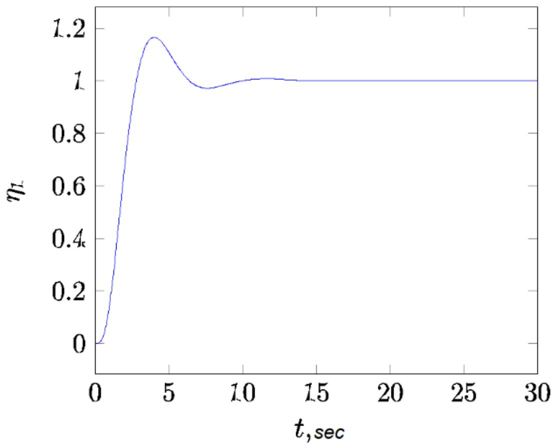

Fig.5.9. Transient processes with gain factor $g(|S|) = |S|^{-1}$

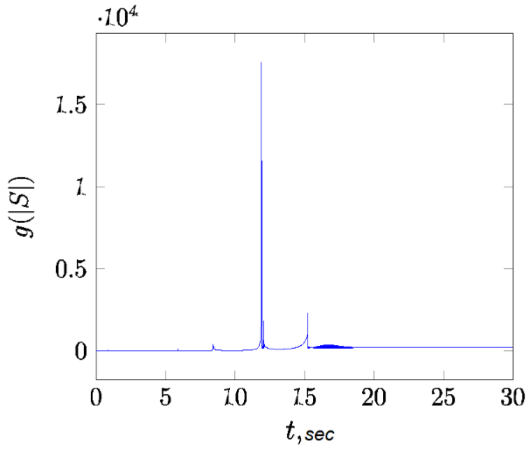

Fig. 5.10. Gain factor $g(|S|) = \dfrac{1}{\sqrt{|S|}}$

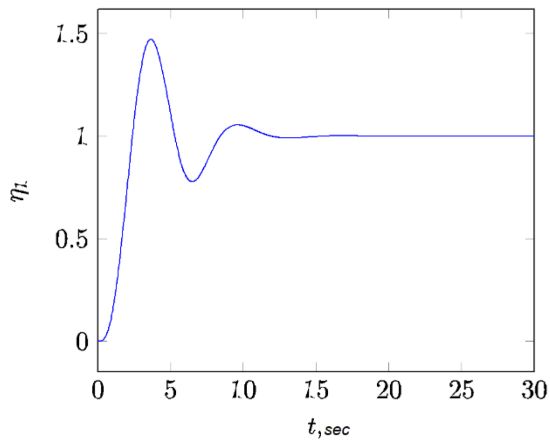

Fig. 5.11. Transient processes with gain factor $g(|S|) = |S|^{-0,5}$

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As can be seen from the provided graphs, the relay system, as the system with the highest gain factor of the controller, exhibits the least overshoot, and therefore the maximum stability margin.

Summarizing the above analysis of the stability of a closed-loop third-order EMS, it can be stated that for the stability of a system with a non-linear activation function it is sufficient that the gain factor $g(|S|)$ obeys the following inequality

$$g(|S|) > \frac{a_2 a_3}{a_1}. \qquad (5.883)$$

Generalizing the condition (5.75) to the case of an arbitrary dynamic system of the n-th order, we can say that the value of the gain factor $g(|S|)$, at which the closed-loop system is stable, is determined from the condition of positivity of the $n-1$ diagonal minor of the Hurwitz determinant

$$M_{(n-1)(n-1)}(a_i, g(|S|)) > 0. \qquad (5.884)$$

he last expression allows determining the minimum permissible values of the gain factor $g(|S|)$ at which the system remains stable. These values for different distributions of the roots of the characteristic equation are shown in Table. 5.1.

Table 5.1. Gain factor values

	Polynomial	Gain factor $g(\|S\|)_{\min}$
Newtonian distribution		
	$p^2 + 2g(\|S\|)\omega_0 p + \omega_0^2$	$g(\|S\|) > 0$
	$p^3 + 3g(\|S\|)\omega_0 p^2 + 3g(\|S\|)\omega_0^2 p + \omega_0^3$	$g(\|S\|) > 1/9$
	$p^4 + 4g(\|S\|)\omega_0 p^3 + 6g(\|S\|)\omega_0^2 p^2 + 4g(\|S\|)\omega_0^3 p + \omega^3$	$g(\|S\|) > 1/5$
Butterworth distribution		
	$p^2 + 1{,}4g(\|S\|)\omega_0 p + g(\|S\|)\omega_0^2$	$g(\|S\|) > 0$

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	$p^3+2g(\lvert S\rvert)\omega_0 p^2+2g(\lvert S\rvert)\omega_0^2 p+g(\lvert S\rvert)\omega_0^3$	$g(\lvert S\rvert)>1/4$
	$p^4+2{,}6g(\lvert S\rvert)\omega_0 p^3+3{,}4g(\lvert S\rvert)\omega_0^2 p^2+2{,}6g(\lvert S\rvert)\omega_0^3 p+g(\lvert S\rvert)\omega^3$	$g(\lvert S\rvert)>0{,}41667$
Distribution implementing sequential regularization based on the modified symmetry principle		
	$p^2+\left(g(\lvert S\rvert)+1\right)\omega_0 p+\omega_0^2 g(\lvert S\rvert)$	$g(\lvert S\rvert)>0$

5.2. Determination of Control Objective

5.2.1. General recommendations for justifying the choice of integral quality functionals

The results obtained in the previous chapter indicate a clear connection between the solution of the problem of analytical design of controllers using the modified symmetry principle and modal control. Therefore, the synthesis of control algorithms as a result of solving optimization problems is of interest.

Let us first consider the general methodology for the mathematical justification of quality functionals that are minimized by optimal controls with a non-linear activation function.

All calculations will be carried out for a generalized dynamic object of the third order, the perturbed motion of which is described by the equations

$$p\eta_1 = b_{11}\eta_1 + b_{12}\eta_2 + b_{13}\eta_3;$$
$$p\eta_2 = b_{21}\eta_1 + b_{22}\eta_2 + b_{23}\eta_3;$$
$$p\eta_3 = b_{31}\eta_1 + b_{32}\eta_2 + b_{33}\eta_3 + m_3 U_3. \qquad (5.885)$$

It should be noted that any electromechanical system can be reduced to a dynamic object of the third order with a known degree of accuracy.

To increase the static accuracy of control systems, we will expand the phase space in which the dynamics of the control object (5.77) is described,

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$$p\eta_0 = \eta_1;$$
$$p\eta_1 = b_{11}\eta_1 + b_{12}\eta_2 + b_{13}\eta_3;$$
$$p\eta_2 = b_{21}\eta_1 + b_{22}\eta_2 + b_{23}\eta_3;$$
$$p\eta_3 = b_{31}\eta_1 + b_{32}\eta_2 + b_{33}\eta_3 + m_3 U_3. \quad (5.886)$$

Generalizing the systems (5.77) and (5.78), we note that here we will consider linear or linearized electromechanical objects with a single input, which are generally described as follows

$$p\eta_i = \sum_{j=1}^{n} b_{ij}\eta_j + m_n U, \; i = 1,...,n \quad (5.887)$$

and

$$p\eta_0 = \eta_k;$$
$$p\eta_i = \sum_{j=1}^{n} b_{ij}\eta_j + m_n U, \quad i = 1,...,n \quad (5.888)$$

where k is the number of the controlled variable, n is the order of the control object.

We introduce a quadratic Lyapunov function into consideration, which determines the dissipation of the stored excess energy on the trajectories of the perturbed motion (5.78)

$$V = \sum_{i,j=0}^{n} V_{ij}\eta_i\eta_j; \quad (5.889)$$

$$V_{ij} = V_{ji}. \quad (5.890)$$

For a dynamic object that is described by equations (5.78), the Lyapunov function (5.81) will be

$$V_1 = V_{00}\eta_0^2 + 2V_{01}\eta_0\eta_1 + 2V_{02}\eta_0\eta_2 + 2V_{03}\eta_0\eta_3 +$$
$$+ V_{11}\eta_1^2 + 2V_{12}\eta_1\eta_2 + 2V_{13}\eta_1\eta_3 + V_{22}\eta_2^2 + 2V_{23}\eta_2\eta_3 + V_{33}\eta_3^2. \quad (5.891)$$

Let us define the quality functional, the minimization of which corresponds to the achievement of the control goal on the trajectories of the perturbed motion (5.78).

This functional will be sought in the class of integral functionals, the integrand of which consists of two terms: one of which depends only on the deviations of the controlled coordinates and determines the dynamic and static characteristics of the control system, and the second, which determines the consumption of control energy, only on the control input.

$$I=\int_0^\infty\left[F(\eta_0,\eta_1,\cdots,\eta_n)+G(U)\right]dt. \tag{5.892}$$

For the object (5.78) and the functional (5.84), replacing the Bellman function with the Lyapunov function, we will compose the basic functional Bellman equation

$$\frac{dV_1}{dt}+F(\eta_0,\eta_1,\eta_2,\eta_3)+G(U)=0, \tag{5.893}$$

where dV_1/dt is the total derivative of the Lyapunov function (5.81)

$$\frac{dV_1}{dt}=\sum_{i=0}^{3}\frac{\partial V_1}{\partial\eta_i}p\eta_i \tag{5.894}$$

or for a generalized dynamic object of the third order (5.78)

$$\begin{aligned}\frac{dV_1}{dt}&=2\left(V_{00}\eta_0+V_{01}\eta_1+V_{02}\eta_2+V_{03}\eta_3\right)\eta_1+\\&+2\left(V_{10}\eta_0+V_{11}\eta_1+V_{12}\eta_2+V_{13}\eta_3\right)\left(b_{11}\eta_1+b_{12}\eta_2+b_{13}\eta_3\right)+\\&+2\left(V_{20}\eta_0+V_{21}\eta_1+V_{22}\eta_2+V_{23}\eta_3\right)\left(b_{21}\eta_1+b_{22}\eta_2+b_{23}\eta_3\right)+\\&+2\left(V_{30}\eta_0+V_{31}\eta_1+V_{32}\eta_2+V_{33}\eta_3\right)\left(b_{31}\eta_1+b_{32}\eta_2+b_{33}\eta_3+m_3U\right)\end{aligned} \tag{5.895}$$

or taking into account the relationship (5.82) between the coefficients V_{ij}

$$\begin{aligned}\frac{dV_1}{dt}&=2\left(V_{00}\eta_0+V_{01}\eta_1+V_{02}\eta_2+V_{03}\eta_3\right)\eta_1+\\&+2\left(V_{01}\eta_0+V_{11}\eta_1+V_{12}\eta_2+V_{13}\eta_3\right)\left(b_{11}\eta_1+b_{12}\eta_2+b_{13}\eta_3\right)+\\&+2\left(V_{02}\eta_0+V_{12}\eta_1+V_{22}\eta_2+V_{23}\eta_3\right)\left(b_{21}\eta_1+b_{22}\eta_2+b_{23}\eta_3\right)+\\&+2\left(V_{03}\eta_0+V_{13}\eta_1+V_{23}\eta_2+V_{33}\eta_3\right)\left(b_{31}\eta_1+b_{32}\eta_2+b_{33}\eta_3+m_3U\right).\end{aligned} \tag{5.896}$$

The total derivative of the Lyapunov function (5.88) fully corresponds to the

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derivative found for a similar linear object and is given in [15].

From the total derivative (5.88), we can extract the total derivative of the Lyapunov function with respect to time

$$\frac{dV_1'}{dt}=2\left(V_{00}\eta_0+V_{01}\eta_1+V_{02}\eta_2+V_{03}\eta_3\right)\eta_1+$$
$$+2\left(V_{01}\eta_0+V_{11}\eta_1+V_{12}\eta_2+V_{13}\eta_3\right)\left(b_{11}\eta_1+b_{12}\eta_2+b_{13}\eta_3\right)+$$
$$+2\left(V_{02}\eta_0+V_{12}\eta_1+V_{22}\eta_2+V_{23}\eta_3\right)\left(b_{21}\eta_1+b_{22}\eta_2+b_{23}\eta_3\right)+$$
$$+2\left(V_{03}\eta_0+V_{13}\eta_1+V_{23}\eta_2+V_{33}\eta_3\right)\left(b_{31}\eta_1+b_{32}\eta_2+b_{33}\eta_3\right) \quad (5.897)$$

for free motion

$$p\eta_0=\eta_1;$$
$$p\eta_1=b_{11}\eta_1+b_{12}\eta_2+b_{13}\eta_3;$$
$$p\eta_2=b_{21}\eta_1+b_{22}\eta_2+b_{23}\eta_3;$$
$$p\eta_3=b_{31}\eta_1+b_{32}\eta_2+b_{33}\eta_3. \quad (5.898)$$

The method of structural-algorithmic synthesis of optimal control systems for electromechanical objects, described in [15], stipulates that this derivative is used to determine the coefficients of the Lyapunov function (5.83) ,as a result of which, on the trajectories of free motion, (5.90) the derivative (5.89) vanishes, i.e.

$$\frac{dV_1'}{dt}=2\left(V_{00}\eta_0+V_{01}\eta_1+V_{02}\eta_2+V_{03}\eta_3\right)\eta_1+$$
$$+2\left(V_{01}\eta_0+V_{11}\eta_1+V_{12}\eta_2+V_{13}\eta_3\right)\left(b_{11}\eta_1+b_{12}\eta_2+b_{13}\eta_3\right)+$$
$$+2\left(V_{02}\eta_0+V_{12}\eta_1+V_{22}\eta_2+V_{23}\eta_3\right)\left(b_{21}\eta_1+b_{22}\eta_2+b_{23}\eta_3\right)+$$
$$+2\left(V_{03}\eta_0+V_{13}\eta_1+V_{23}\eta_2+V_{33}\eta_3\right)\left(b_{31}\eta_1+b_{32}\eta_2+b_{33}\eta_3\right)=0. \quad (5.899)$$

Substituting the value of the derivative (5.91) into the Bellman equation (5.85), we simplify the equation (5.85) and present it in expanded form

$$2\left(V_{30}\eta_0+V_{31}\eta_1+V_{32}\eta_2+V_{33}\eta_3\right)m_3U+$$
$$+F(\eta_0,\eta_1,\eta_2,\eta_3)+G(U)=0 \quad (5.900)$$

or

$$SU + F(\eta_0, \eta_1, \eta_2, \eta_3) + G(U) = 0, \qquad (5.901)$$

where

$$S = 2m_3\left(V_{30}\eta_0 + V_{31}\eta_1 + V_{32}\eta_2 + V_{33}\eta_3\right). \qquad (5.902)$$

Expression S will be called the equilibrium equation of the system.

For an electromechanical control object of arbitrary order n, the dynamics of which are described by the equations of perturbed motion (5.80), the equation (5.94) will take the following form

$$S = 2m_n \sum_{i=0}^{n} V_{in}\eta_i. \qquad (5.903)$$

Differentiation of the equation (5.92) with respect to the control input U allows us to obtain the dependence

$$2m_3\left(V_{30}\eta_0 + V_{31}\eta_1 + V_{32}\eta_2 + V_{33}\eta_3\right) = -g(U) \qquad (5.904)$$

or taking into account the expression (5.94)

$$S = -g(U), \qquad (5.905)$$

цhere

$$g(U) = \frac{\partial G(U)}{\partial U}. \qquad (5.906)$$

The desired optimal control can be found as a result of solving, in the general case, a тon-linear equation (5.96)

$$U = -f\left[2m_3\left(V_{30}\eta_0 + V_{31}\eta_1 + V_{32}\eta_2 + V_{33}\eta_3\right)\right], \qquad (5.907)$$

or in general terms

$$U = -f\left(S\right) = -f(|S|)\operatorname{sign}(S), \qquad (5.908)$$

where function $f(.)$ is an odd function of its argument

$$f(-S) = -f(S) = f(|S|)\operatorname{sign}(S), \qquad (5.909)$$

Thus, the synthesis of optimal controls with a тon-linear activation function can be conditionally divided into two stages: determining the coefficients of the Lyapunov function (5.83) through the parameters of the electromechanical object and determining such a тon-linear activation function that corresponds to the condition

$$f\big(g(U)\big)=U, \tag{5.910}$$

i.e. functions $f(S)$ and $g(U)$ are inverse to each other

$$f(S)=-U=-g^{-1}(U),\; g(U)=-S=-f^{-1}(S), \tag{5.911}$$

which follows from the analysis of equations (5.97) and (5.99), generalized to equations (5.103).

The use of dependencies (5.103) allows us to uniquely link the optimal control algorithm given by the expression (5.100) and the second term of the functional (5.84)

$$G(U)=\int g(U)dU=-\int f^{-1}(S)dU. \tag{5.912}$$

Differentiating the first equation of the system (5.103) with respect to time

$$\frac{dU}{dt}=-\frac{\partial f(S)}{\partial S}\frac{dS}{dt}, \tag{5.913}$$

we can define the differential dU

$$dU=-\frac{\partial f(S)}{\partial S}dS \tag{5.914}$$

and, substituting it into the integral (5.104), we get

$$G(U)=\int g(U)dU=\int f^{-1}(S)\frac{\partial f(S)}{\partial S}dS=\int S\frac{\partial f(S)}{\partial S}dS. \tag{5.915}$$

Let us now move on to the definition of the first term of the functional (5.84), which can be found from the equation (5.92) by substituting the value of the integral (5.104) and the control input (5.99) into it.

To determine this component of the functional, we substitute the values of the expressions (5.94), (5.104), (5.100) into the equation (5.92)

$$-Sf(S)+F(\eta_0,\eta_1,\eta_2,\eta_3)+\int S\frac{\partial f(S)}{\partial S}dS=0. \tag{5.916}$$

It clearly follows from the equation (5.108) that

$$F(\eta_0, \eta_1, \eta_2, \eta_3) = Sf(S) - \int S \frac{\partial f(S)}{\partial S} dS. \tag{5.917}$$

It is easy to see that substituting the value of the function that determines the dispersion of accumulated excess energy on the trajectories of the perturbed motion (5.109) and the expression that determines the control energy consumption (5.104) into the functional (5.84), allows us to obtain the following quality functional

$$I = \int_0^{\infty} \left[\left(Sf(S) - \int S \frac{\partial f(S)}{\partial S} dS \right) + \int S \frac{\partial f(S)}{\partial S} dS \right] dt =$$
$$= \int_0^{\infty} Sf(S) dt, \tag{5.918}$$

which can be interpreted as the energy that is dissipated during the control of an electromechanical object (5.80).

However, using the quality functional found this way does not allow us to obtain an unambiguous solution to the Bellman equation (5.85) due to the absence of a control input in the integrand of the quality functional.

A similar situation arises in the synthesis of optimal systems with discontinuous control of the form

$$U = -\text{sign}(S) \tag{5.919}$$

for which the first equation of the system (5.103) will be

$$f(S) = -U = \text{sign}(S). \tag{5.920}$$

It is obvious that substituting a function (5.112) into a functional (5.110) allows us to obtain a known functional used in creating relay systems.

$$I = \int_0^{\infty} Sf(S) dt =$$
$$= \int_0^{\infty} S \cdot \text{sign}(S) dt = \int_0^{\infty} |S| dt. \tag{5.921}$$

However, despite the identity of the functionals (5.110), (5.113) and functional

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$$I = \int_0^\infty \left[Sf(S) - \int S \frac{\partial f(S)}{\partial S} dS + \int g(U) dU \right] dt \tag{5.922}$$

from a methodological point of view, it is preferable to use the functional (5.114) as the control goal, as it explicitly highlights the energy components characterizing the control process.

5.2.2. Determination of the quality functional for linear optimal control systems

Before moving on to defining control objectives for objects whose dynamics is represented by equations different from (5.77), and considering particular cases of using various activation functions, we will show that the obtained functionals are a generalization for a wide class of control objectives in relay and linear systems and, in the case of linear controls, correspond to the well-known Letov functionals [28].

Let us use the results given in [15], and consider the control input

$$U = -\frac{1}{2C} S, \tag{5.923}$$

under the action of which a dynamic object from any initial position moves to the origin along exponential trajectories. The linearity of such control is evident. Consequently, the conjugate functions $f(S), g(U)$ dividing the plane SU into two half-planes in the ratio $1:2C$ (Fig. 5.12), can be represented as follows in accordance with expressions (5.115) and (5.103)

$$\begin{aligned} f(S) &= -U = \frac{1}{2C} S; \\ g(U) &= -S = 2CU. \end{aligned} \tag{5.924}$$

Then, the function that determines the quality of transient processes in an electromechanical system

$$F(\eta_0, \eta_1, \eta_2, \eta_3) = S\frac{1}{2c}S - \int S\frac{1}{2c}dS = \frac{1}{4c}S^2. \tag{5.925}$$

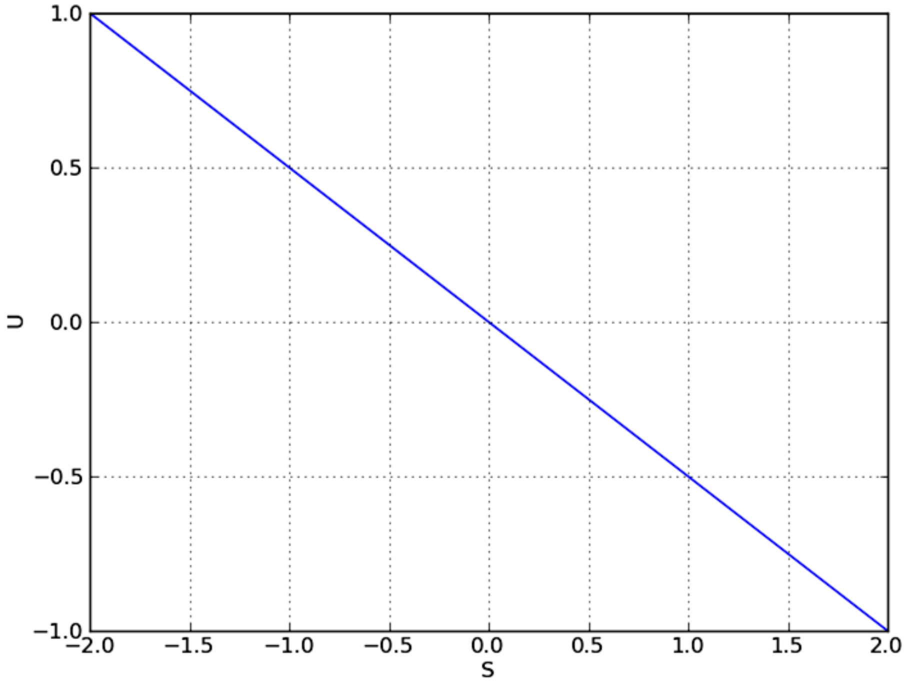

Fig. 5.12. Division of a plane SU into two half-planes $C = 1$

Integrating the last equation of the conjugate system (5.116) allows us to determine the component reflecting the control energy spent on the transient process

$$G(U) = \int g(U)dU = \int 2CUdU = cU^2. \tag{5.926}$$

Taking into account the expressions (5.117) and (5.118) the desired functional will take the following form

$$I = \int_0^\infty \left[\frac{1}{4c}S^2 + cU^2 \right] dt. \tag{5.927}$$

Taking into account the value of the expression (5.94), we write the functional (5.119) in the form

$$I = \int_0^\infty \left[\frac{m_3^2}{c}(V_{30}\eta_0 + V_{31}\eta_1 + V_{32}\eta_2 + V_{33}\eta_3)^2 + cU^2 \right] dt. \tag{5.928}$$

Graphically, the functional (5.120) represents a curve changing according to exponential laws. An example of such a curve is shown in Fig. 5.13.

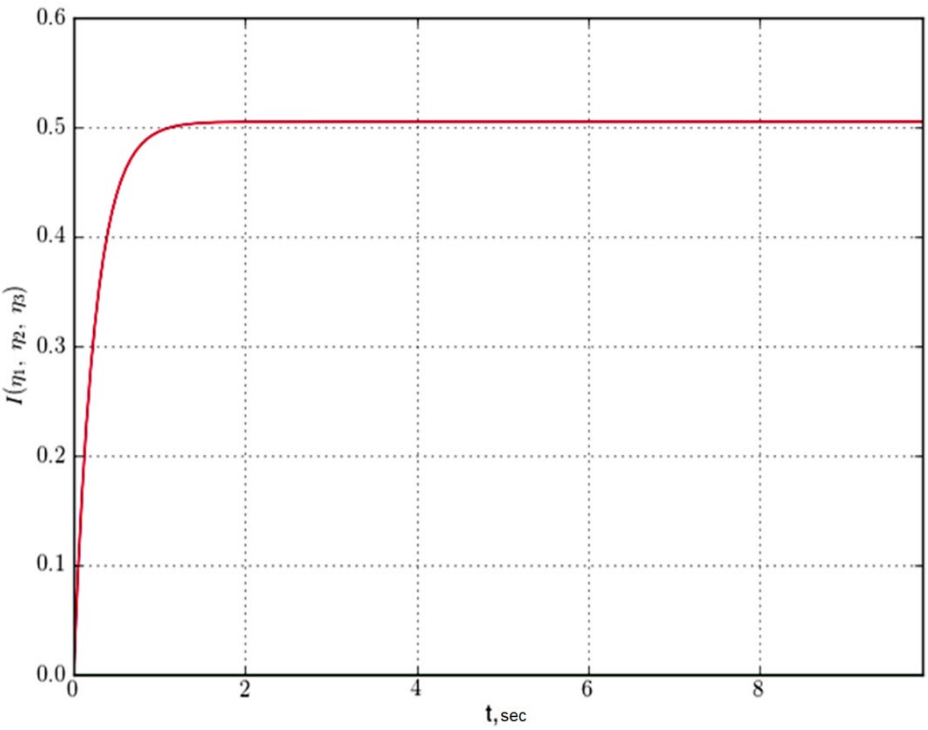

Fig. 5.13. Trajectories of change in the quality criterion of control processes

Comparison of the functional defined in this way with the known results shows its complete identity with the known functional [15].

This allows us to conclude that the functional (5.120) is a special case of the functional (5.114) under the condition of linearity of the conjugate functions (5.116).

The obtained confirmation of the correctness of the proposed approach allows us to proceed to the consideration of control systems with non-linear activation functions and the definition of control objectives for them.

5.2.3. Construction of integral quality functionals characterizing control processes of linear systems in closed phase spaces

Currently, in the theory of optimal control, integral quality functionals, which are variations of the Letov functional, are used in the synthesis of linear control systems. These functionals are defined for an open domain of the space of state variables and, accordingly, their minimization is carried out by optimal controls that are not limited in modulus.

To take into account natural limitations on the control signal, a limitation of the output signal is forcibly introduced into the synthesized optimal algorithms [15].

However, the controls defined in this way minimize the original quality functionals only in the linear part.

Therefore, as an example of using the methodology given in the previous section, we will consider the definition of the control objective, which is achieved on the trajectories of the perturbed motion (5.78) by optimal control (5.115), which is limited in modulus

$$U = -\mathrm{sat}\left[\frac{1}{2C}S\right]. \tag{5.929}$$

where $\mathrm{sat}(.)$ is a "saturation" type function

$$\mathrm{sat}(x) = \begin{cases} 1, & x > 1, \\ -1, & x < -1, \\ x, & -1 \le x \le 1 \end{cases} = \begin{cases} \mathrm{sign}(x), & if \quad |x| > 1, \\ x, & if \quad |x| \le 1 \end{cases} \tag{5.930}$$

Graphically the function (5.122) looks as follows:

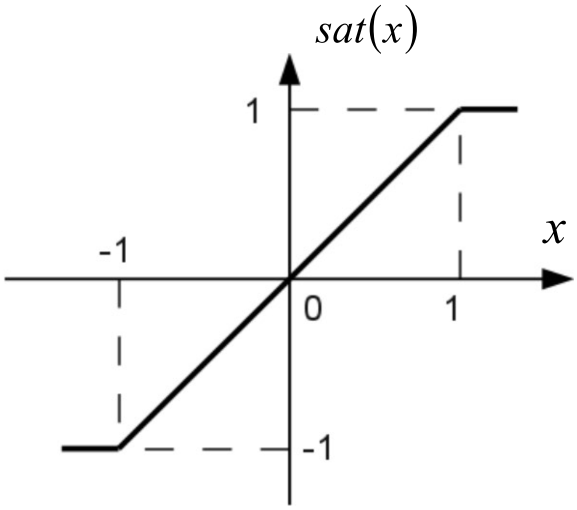

Fig. 5.14. The sat(.) function

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Taking into account the expression (5.122) and control (5.121), we define the conjugate functions $g(U)$ and $f(S)$ as follows:

$$U = -f(S) = -\mathrm{sat}\left[\frac{1}{2C}S\right],$$
$$S = -g(U) = -\mathrm{sat}^{-1}(U), \qquad (5.931)$$

where

$$\mathrm{sat}^{-1}(x) = \begin{cases} 1\ldots+\infty, & x=1, \\ -1\ldots-\infty, & x=-1, \\ x, & |x| \le 1. \end{cases} \qquad (5.932)$$

A graphical representation of the function (5.124) is shown in Fig. 5.15.

The integral of a function (5.124) is determined by analogy with polynomial functions of odd degrees

$$\int sat^{-1}(x)dx = \begin{cases} 1, & if \quad |x| = 1, \\ x^2/2, & if \quad |x| \le 1. \end{cases} \qquad (5.933)$$

Graphically the function (5.125) looks as shown in Fig. 5.16.

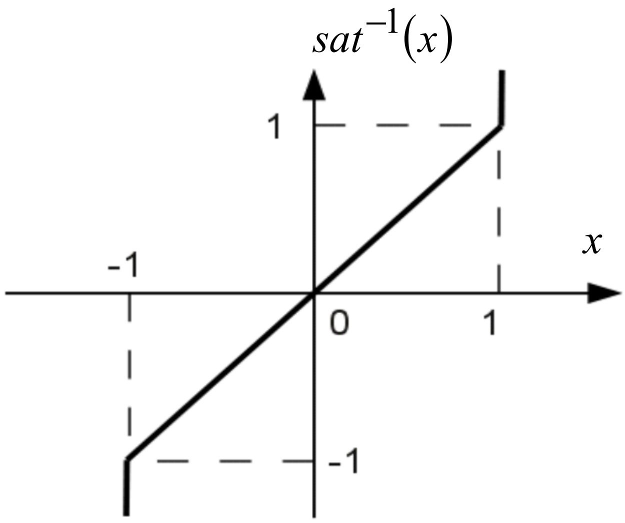

Fig. 5.15. The function $\mathrm{sat}^{-1}(.)$

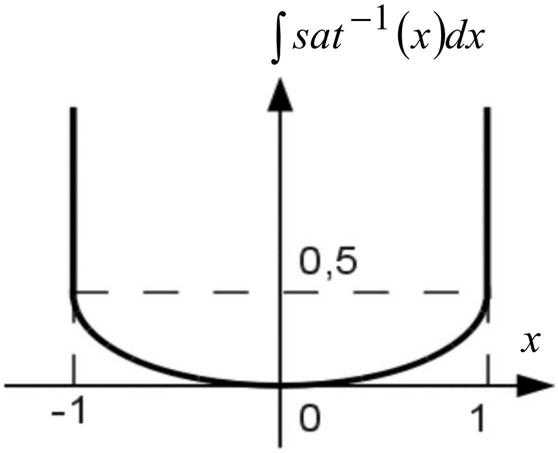

Fig. 5.16. The function $\int \mathrm{sat}^{-1}(x)dx$ -

Then, taking into account the expression (5.125) and using the dependencies (5.107) and (5.109) found in the previous section, we determine the components of the quality functional, which is minimized on the trajectories of the perturbed motion of the system (5.78) by optimal control (5.121)

$$G(U)=\int g(U)du=\int \mathrm{sat}^{-1}(U)dU=\begin{cases} 1, & if \ \ |U|\geq 1, \\ U^2/2, & if \ \ |U|<1. \end{cases} \quad (5.934)$$

and

$$F(\eta_0,\eta_1,\eta_2,\eta_3)=S\cdot \mathrm{sat}\left[\frac{1}{2C}S\right]-\int S\frac{\partial}{\partial S}\mathrm{sat}\left[\frac{1}{2C}S\right]dS=$$

$$=\begin{cases} |S|, & |S|>1, \\ 0,5\cdot S^2/C, & |S|\leq 1 \end{cases} -\int \begin{cases} 0, & |S|>1, \\ \dfrac{S}{4C}, & |S|\leq 1 \end{cases} dS=$$

$$=\begin{cases} |S|, & |S|>1, \\ 0,5\cdot S^2/C, & |S|\leq 1 \end{cases} -\begin{cases} 0, & |S|>1, \\ \dfrac{S^2}{8C}, & |S|\leq 1 \end{cases} =$$

$$=\begin{cases} |S|, & |S|>1, \\ \dfrac{S^2}{8C}, & |S|\leq 1. \end{cases} \quad (5.935)$$

Substituting the components (5.126) and (5.127) into the functional (5.84) allows us to determine the following type of functional, the minimization of which is carried out by optimal control with a piecewise-continuous activation function (5.122)

$$I = \int_0^{\infty} \left[\begin{cases} |S|, & |S| > 1, \\ 0{,}125 \cdot S^2/C, & |S| \le 1. \end{cases} + \begin{cases} 1, & if \quad |U| = 1 \\ U^2/2, & if \quad |U| \le 1 \end{cases} \right] dt. \tag{5.936}$$

Taking into account dependencies (5.122), (5.123) and (5.124), we represent the functional (5.128) as follows

$$I = \int_0^{\infty} \left[\begin{cases} |S|, & |S| > 1, \\ 0{,}125 \cdot S^2/C, & |S| \le 1. \end{cases} + \begin{cases} 1, & if \quad |S| > 1, \\ U^2/2, & if \quad |S| \le 1. \end{cases} \right] dt \tag{5.937}$$

or

$$I = \int_0^{\infty} \left[\begin{cases} |S| + 1, & if \ |S| > 1, \\ 0{,}125 \cdot S^2/C + U^2/2, & if \ |S| \le 1. \end{cases} \right] dt \tag{5.938}$$

An interesting feature of functionals (5.129) and (5.130) is the variability of their structure, which allows these functionals to be represented as two functionals

$$I_1 = \int_0^{\infty} [|S| + C] dt, \ C = 1;$$
$$I_2 = \int_0^{\infty} \left[0{,}125 \cdot S^2/C + U^2/2 \right] dt =$$
$$= \int_0^{\infty} \left[0{,}125 \cdot |S|^2 /C + |U|^2 /2 \right] dt, \tag{5.939}$$

combined by the switching rule

$$I = \begin{cases} I_1, & |S| > 1, \\ I_2, & |S| \le 1. \end{cases} \tag{5.940}$$

From the analysis of the expressions (5.131) it follows that for large deviations from the equilibrium line S, the optimal control minimizes the integral quality functional I_1 on the trajectories of the perturbed motion of the system (5.78).

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Considering that similar results were obtained earlier for relay control systems [15], we can make a conclusion about the motion under the action of the maximum possible control input

$$U = -\operatorname{sign}\left[\frac{1}{2C} \cdot S\right] \quad (5.941)$$

along trajectories

$$S = 2m_n \sum_{i=0}^{n} V_{in} \eta_i > 1, \quad (5.942)$$

The correctness of this statement is confirmed by the analysis of the expression (5.122).

As the system approaches the equilibrium position, the equation of the equilibrium line (5.95) becomes true for the relation

$$S = 2m_3 \sum_{i=0}^{3} V_{ni} \eta_i \leq 1, \quad (5.943)$$

and this means that the controller implementing the law (5.121) exits saturation, and the further movement of the control system with the generalized electromechanical object (5.78) occurs under the condition of minimization of the functional I_2.

Thus, in the presence of a constraint on the module of the control input, the quality of the control processes is characterized by the values of the functionals I_1 and I_2. The transition from one functional to another occurs when the equilibrium line reaches the constraint value

$$|S| = 1, \quad (5.944)$$

Moreover, since this magnitude equals to a conventional unit, the transition from one functional to another is not accompanied by discontinuities, and therefore cannot serve as a reason for increasing the oscillation of the control system.

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5.2.4. Determination of quality functionals characterizing control processes in systems with "Square root with sign consideration" activation function

To clarify the general patterns and features of the synthesis of integral quality functionals for non-linear optimal control systems with an irrational activation function, we first consider an optimal control system with a non-linear activation function of the "square root" type.

As a control object we will consider a generalized third-order dynamic object, the dynamics of which in the extended phase space is described by differential equations (5.78).

In the publication [39], it is proposed to perform optimal control of such an object in accordance with the algorithm

$$U = -\mathrm{sqrt}\left[2m_3\left(V_{30}\eta_0 + V_{31}\eta_1 + V_{32}\eta_2 + V_{33}\eta_3\right)\right] \quad (5.945)$$

or taking into account the expression (5.94)

$$U = -\mathrm{sqrt}(S), \quad (5.946)$$

where $\mathrm{sqrt}(.)$ is a function of the type " sign-adjusted square root"

$$\mathrm{sqrt}(x) = \begin{cases} \mathrm{sign}(x), & |x| > 1, \\ \sqrt{|x|}\,\mathrm{sign}(x), & |x| \le 1. \end{cases} \quad (5.947)$$

A graphical representation of the function (5.139) is shown in Fig. 5.17.

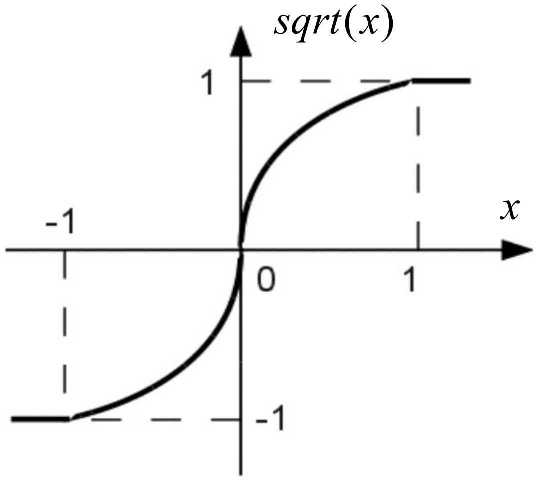

Fig. 5.17. The sqrt(.) function

Taking into account the expression (5.139) and control (5.138), we define the conjugate functions $g(U)$ and $f(S)$ as follows:

$$\begin{aligned} U &= -f(S) = -\mathrm{sqrt}(S), \\ S &= -g(U) = -\mathrm{sqr}(U), \end{aligned} \qquad (5.948)$$

where

$$\mathrm{sqr}(x) = \begin{cases} 1, & if \quad x = 1, \\ -1, & if \quad x = -1, \\ x^2 \,\mathrm{sign}(x), & if \quad |x| \le 1. \end{cases} \qquad (5.949)$$

Graphically, the function (5.141) looks as shown in Fig. 5.18.

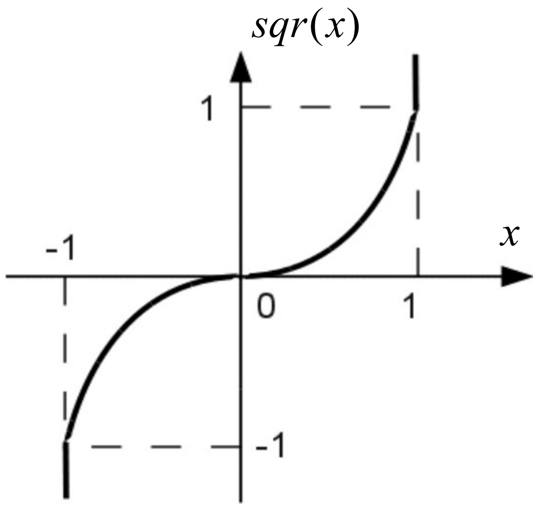

Fig. 5.18. The sqr(.) function

Considering that

$$\int \mathrm{sign}(x)dx = |x|, \qquad (5.950)$$

the integral of the function (5.141) is determined by analogy with polynomial functions of odd degrees

$$\int \mathrm{sqr}(x)dx = \begin{cases} 1\ldots+\infty, & if \quad x = 1, \\ 1\ldots+\infty, & if \quad x = -1, \\ |x|^3/3, & if \quad |x| \le 1. \end{cases} \qquad (5.951)$$

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Then, taking into account the expressions (5.139)–(5.141), (5.143) and using the dependencies (5.107) and (5.109), we determine the components of the quality functional, which is minimized on the trajectories of the perturbed motion of the object (5.78) with optimal control (5.137)

$$G(U)=\int g(U)du=\int \mathrm{sqr}(U)dU=\begin{cases} 1, & if \ \ |U|=1, \\ |U|^3/3, & if \ \ |U|\le 1. \end{cases} \qquad (5.952)$$

and

$$\begin{aligned} F(\eta_0,\eta_1,\eta_2,\eta_3) &= S\cdot sqrt(S)-\int S\frac{\partial}{\partial S}sqrt(S)dS= \\ &=\begin{cases} |S|, & |S|>1, \\ \sqrt{|S|^3}, & |S|\le 1 \end{cases}-\int\begin{cases} 0, & |S|\ge 1, \\ 1/2\sqrt{|S|}, & |S|<1 \end{cases}dS= \\ &=\begin{cases} |S|, & |S|>1, \\ \sqrt{|S|^3}, & |S|\le 1 \end{cases}-\begin{cases} 0, & |S|\ge 1, \\ 1/3\sqrt{|S|^3}, & |S|<1 \end{cases}= \\ &=\begin{cases} |S|, & |S|\ge 1, \\ 2/3\sqrt{|S|^3}, & |S|<1. \end{cases} \end{aligned} \qquad (5.953)$$

Substituting the components (5.144) and (5.145) into the integral (5.84) allows us to determine the type of functional, the minimization of which is carried out by optimal control with a non-linear activation function (5.139),

$$I=\int_0^\infty\left[\begin{cases} |S|, & |S|>1, \\ 2/3\sqrt{|S|^3}, & |S|\le 1. \end{cases}+\begin{cases} 1, & if \ \ |U|=1, \\ |U|^3/3, & if \ \ |U|\le 1. \end{cases}\right]dt. \qquad (5.954)$$

Taking into account the dependencies (5.139) and (5.140), we represent the functional (5.146) as follows

$$I=\int_0^\infty\left[\begin{cases} |S|, & |S|>1, \\ 2/3\sqrt{|S|^3}, & |S|\le 1. \end{cases}+\begin{cases} 1, & if \ \ |S|\ge 1 \\ |U|^3/3, & if \ \ |S|<1 \end{cases}\right]dt \qquad (5.955)$$

or

$$I=\int_0^\infty\begin{cases} |S|+1, & if \ \ |S|\ge 1 \\ 2/3\sqrt{|S|^3}+|U|^3/3, & if \ \ |S|<1 \end{cases}dt \qquad (5.956)$$

Similar to the functionals (5.129) and (5.130), which were defined in the previous section, the functionals (5.147) and (5.148) can also be classified as functionals with a variable structure, allowing them to be represented as two functionals

$$I_1 = \int_0^{\infty} [|S| + C] dt, C = 1; \quad I_2 = \int_0^{\infty} \left[2/3\sqrt{|S|^3} + |U|^3/3 \right] dt, \tag{5.957}$$

combined by the following switching rule

$$I = \begin{cases} I_1, & |S| \geq 1, \\ I_2, & |S| < 1. \end{cases} \tag{5.958}$$

From the analysis of expressions (5.149), it follows that for large deviations from the equilibrium line S, the optimal control minimizes the integral quality functional (5.78) on the trajectories of the perturbed motion I_1. Considering that this functional was obtained earlier for relay control systems [15], we can conclude that the motion is under the influence of the maximum possible control input

$$U = -\text{sign}(S). \tag{5.959}$$

The correctness of this statement is confirmed by the analysis of the expression (5.139).

Analysis of functionals (5.147) and (5.148) and their comparison in the domain $|S| < 1$ with functionals (5.129) and (5.130) allows us to conclude that in control systems with a square root-type activation function, the absolute values of the functionals are greater compared to linear systems.

The above creates the preconditions for a comprehensive consideration of control systems with irrational activation functions.

5.2.5. Determination of quality functionals characterizing control processes in systems with an arbitrary order irrational activation function

For the generalized electromechanical control object (5.78), the optimal control can be represented as follows:

$$U = -f_1(S), \tag{5.960}$$

where

$$f_1(S) = \begin{cases} 1, & if \quad S > 1, \\ -1, & if \quad S < -1, \\ |S|^{\alpha}\,\mathrm{sign}(S), & if \quad -1 \le S \le 1, \end{cases} \tag{5.961}$$

here α is any real exponent.

As follows from comparison of functions (5.139) and (5.141), the function conjugate to the function (5.153) is

$$f_2(U) = \begin{cases} 1, & if \quad U = 1, \\ -1, & if \quad U = -1, \\ U^{1/\alpha}\,\mathrm{sign}(U), & if \quad |U| \le 1, \end{cases} \tag{5.962}$$

and its integral can be represented as follows

$$\int f_2(U)dU = \begin{cases} 1 \cdots + \infty, & |U| = 1, \\ |U|^{1+1/\alpha} / (1 + 1/\alpha), & |U| \le 1, \end{cases} \tag{5.963}$$

Taking into account the expression (5.153) and control (5.152), we define the conjugate functions $g(U)$ and $f(S)$ as follows:

$$U = -f_1(S) = -\begin{cases} \mathrm{sign}(S), & if \quad |S| \ge 1, \\ |S|^{\alpha} \cdot \mathrm{sign}(S), & if \quad -1 < S < 1, \end{cases}$$

$$S = -g(U) = -f_2(U) = \begin{cases} (1 \cdots + \infty)\,\mathrm{sign}(U), & if \quad |U| = 1, \\ U^{1/\alpha}\,\mathrm{sign}(U), & if \quad |U| \le 1, \end{cases} \tag{5.964}$$

Then, taking into account the expressions (5.153), (5.154), (5.155) and using the dependencies (5.107) and (5.109), we determine the components of the quality functional, which is minimized on the trajectories of the perturbed motion (5.78) by the optimal control (5.152)

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$$G(U)=\int g(U)du=\int f_2(U)dU=\begin{cases}1, & |U|=1,\\ |U|^{\frac{1+\alpha}{\alpha}}/(1+1/\alpha), & |U|\le 1.\end{cases} \quad (5.965)$$

and

$$\begin{aligned}F(\eta_0,\eta_1,\eta_2,\eta_3)&=S\cdot f_1(S)-\int S\frac{\partial}{\partial S}f_1(S)dS=\\&=\begin{cases}|S|, & |S|>1,\\ |S|^{1+\alpha}, & |S|\le 1\end{cases}-\int\begin{cases}0, & |S|\ge 1,\\ \alpha|S|^{\alpha}, & |S|<1\end{cases}dS=\\&=\begin{cases}|S|, & |S|>1,\\ |S|^{1+\alpha}, & |S|\le 1\end{cases}-\begin{cases}0, & |S|\ge 1,\\ \alpha/(1+\alpha)|S|^{1+\alpha}, & |S|<1\end{cases}=\\&=\begin{cases}|S|, & |S|>1,\\ |S|^{1+\alpha}/(1+\alpha), & |S|\le 1.\end{cases}\end{aligned} \quad (5.966)$$

Substituting the components (5.157) and (5.158) into the functional (5.84) allows us to determine the following type of functional, the minimization of which is carried out by optimal control with an irrational activation function (5.153)

$$I=\int_0^{\infty}\left[\begin{cases}|S|, & |S|\ge 1,\\ |S|^{1+\alpha}/(1+\alpha), & |S|<1.\end{cases}+\begin{cases}1, & |U|=1,\\ |U|^{1+1/\alpha}/(1+1/\alpha), & |U|\le 1.\end{cases}\right]dt. \quad (5.967)$$

Taking into account the dependencies (5.153) and (5.156), we represent the functional (5.159) as follows

$$I=\int_0^{\infty}\left[\begin{cases}|S|, & |S|\ge 1,\\ |S|^{1+\alpha}/(1+\alpha), & |S|<1.\end{cases}+\begin{cases}1, & |S|\ge 1,\\ |U|^{1+1/\alpha}/(1+1/\alpha), & |S|<1.\end{cases}\right]dt \quad (5.968)$$

or

$$I=\int_0^{\infty}\left[\begin{cases}|S|+1, & |S|\ge 1,\\ |S|^{1+\alpha}/(1+\alpha)+|U|^{1+1/\alpha}/(1+1/\alpha), & |S|<1.\end{cases}\right]dt. \quad (5.969)$$

Similar to the previously considered cases for control systems with activation functions of the "saturation" and "square root with sign" types, the components whose minimization is ensured by the controller's operation in different zones can be explicitly identified in the functional (5.161), and the functional itself can be reduced to the form (5.150). However, due to its obviousness, this form of the functional will not be considered.

Instead, let us pay attention to the coefficient α, which determines the type of activation function. Among all possible values of α, we will highlight several characteristic points and segments:

- $\alpha \in (-\infty, 0)$ – this type of systems containing a complex hyperbolic activation function and requires further investigation.

Both the study of the static and dynamic properties of systems with hyperbolic activation functions and the determination of extremals of the minimized quality functional are of interest.

- $\alpha = 0$ – this value of α uniquely defines the relay control system, which, according to the algorithm (5.133) optimizes the functional along the trajectories of the perturbed motion of the electric drive (5.90)

$$I = \lim_{\alpha \to 0} \int_0^{\infty} \left[\begin{cases} |S| + 1, & |S| \geq 1, \\ |S|^{1+\alpha} / (1+\alpha) + |U|^{\frac{1+\alpha}{\alpha}} / (1 + 1/\alpha), & |S| < 1. \end{cases} \right] dt, \tag{5.970}$$

which, taking into account the constraint on the control signal (5.133), can be simplified as follows

$$I = \int_0^{\infty} \left[\begin{cases} |S| + 1, & |S| \geq 1, \\ |S|, & |S| < 1. \end{cases} \right] dt. \tag{5.971}$$

- $\alpha \in (0, 1)$ – this value of α defines a control system with an irrational activation function. Moreover, such a system occupies an intermediate position between systems with relay and linear controls in terms of its characteristics[40].

- $\alpha = 1$ – this value of α defines a linear control system, which, in accordance

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with the algorithm (5.121), optimizes the functional on the trajectories of the perturbed motion of the electric drive (5.90)

$$I = \int_0^{\infty} \left[\begin{cases} |S| + (1 \ldots + \infty), & |S| > 1, \\ |S|^2/2 + |U|^2/2, & |S| \le 1. \end{cases} \right] dt, \tag{5.972}$$

which, up to the weighting coefficients, corresponds to the previously found one (5.130)

- $\alpha \in (1, +\infty)$ – specific solutions for the synthesis and use of quadratic and other controllers [41] are known and require further investigation.

5.2.6. Construction of functionals of additive form for control systems with irrational activation function

The use of quality functionals, discussed in the previous sections for the synthesis of optimal systems makes it possible to satisfy a wide range of control objectives for numerous technological processes and production systems.

However, the use of non-linear functions (5.139) and (5.153) does not always allow the formation of the desired trajectories of the perturbed motion due to the finite range of values of these non-linearities. Therefore, to solve the problem of forming the desired trajectory, we will generalize the results obtained in the previous sections and look for control inputs that ensure the achievement of the control objective, in the following form

$$U_{\Sigma} = -\mathrm{sat}\left[\sum_{i=1}^{m} w_i f_i(S) \right] \tag{5.973}$$

where $f_i(S)$ are alternating irrational functions depending on the coefficient α ; w_i are some weighting coefficients.

Let us find the quality functional that is minimized by control (5.165) on the trajectories of motion (5.77).

In an open domain, control (5.165) can be represented as follows:

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$$U_{\Sigma}=-\sum_{i=1}^{m}w_i f_i(S)=\sum_{i=1}^{m}w_i U_i. \tag{5.974}$$

We will assume that each i-th term of the control input (5.166) in the open domain minimizes the functional (5.84), which in this case takes the form

$$I_i=\int_0^{\infty}[F_i(S)+G_i(U)]dt, \tag{5.975}$$

where the unknown functions $F_i(S)$ and $G_i(U)$ are determined in accordance with the methodology discussed in the previous sections.

Analysis of the algorithm (5.166) allows us to represent the functional minimized in the open domain as follows

$$I=\sum_{i=1}^{m}w_i I_i=\int_0^{\infty}\sum_{i=1}^{m}w_i[F_i(S)+G_i(U)]dt. \tag{5.976}$$

In a closed domain (5.168), the functional will take a form similar to (5.130)

$$I=\int_0^{\infty}\left[\begin{cases}\sum_{i=1}^{m}w_i|f_i(S)|+(1\ldots+\infty), & \sum_{i=1}^{m}w_i|f_i(S)|>1,\\ \sum_{i=1}^{m}w_i\left[F_i(S)+G_i(U)\right], & \sum_{i=1}^{m}w_i|f_i(S)|\le 1.\end{cases}\right]dt, \tag{5.977}$$

or

$$I=\int_0^{\infty}\left[\begin{cases}\sum_{i=1}^{m}w_i\,|\,f_i(S)|+(1\ldots+\infty), & |S|>|S_{gr}|,\\ \sum_{i=1}^{m}w_i[F_i(S)+G_i(U)], & |S|\le|S_{gr}|,\end{cases}\right]dt, \tag{5.978}$$

where S_{gr} is the coordinate of the representative point on the switching line where the control system is closed-loop. In general case, this coordinate is determined by solving the equation

$$\sum_{i=1}^{m} w_i \mid f_i\left(S_{gr}\right) \mid = 1. \tag{5.979}$$

As an example, we define a functional that is minimized on the trajectories of the perturbed motion (5.77) by the control input

$$U = -\mathrm{sat}\left[\frac{1}{2C}\left(S + \sqrt{|S|}\,\mathrm{sign}[S]\right)\right]. \tag{5.980}$$

The first control term (5.172) minimizes the functional (5.131), the second, taking into account the expression (5.145), subject to the constraint $|S| \leq 1$, delivers a minimum to the functional

$$I = \frac{1}{6C}\int_0^{\infty}\left(2\sqrt{|S|^3} + |U|^3\right)dt. \tag{5.981}$$

Algebraic summation of functionals (5.131) and (5.173) allows us to write the following quality functional

$$I = \frac{1}{2C}\int_0^{\infty}\left[\frac{1}{3}\left(2\sqrt{|S|^3} + |U|^3\right) + \frac{1}{2}\left(|S|^2 + 2|U|^2\right)\right]dt \tag{5.982}$$

or

$$I = \frac{1}{2C}\int_0^{\infty}\left[\left(\frac{2}{3}\sqrt{|S|^3} + \frac{1}{2}|S|^2\right) + \left(\frac{1}{3}|U|^3 + |U|^2\right)\right]dt \quad , \tag{5.983}$$

which is minimized by control (5.172) in the open domain.

Before moving on to defining the quality functional that is minimized by control (5.165), we will solve the equation (5.171) and determine the position of the representative point at which the control system opens, accompanying the controller entering saturation.

For the control input (5.172), equation (5.171) will have the following form

$$\left(|S_{gr}| + \sqrt{|S_{gr}|}\right) = 2C. \tag{5.984}$$

Solution of the equation (5.176) with respect to $|S_{gr}|$

$$|S_{gr}| = \sqrt{0,5\sqrt{8C+1} - 0,5} \tag{5.985}$$

allows us to write the control as follows

$$U = -\begin{cases} \text{sign}\left(S + \sqrt{|S|}\,\text{sign}[S]\right), & |S| > |S_{gr}|, \\ S + \sqrt{|S|}\,\text{sign}[S], & |S| \le |S_{gr}|. \end{cases} \tag{5.986}$$

Taking into account that the control (5.178) is a piecewise-continuous function containing two zones, the integral quality functional minimized by this control will be sought in the form

$$I = \begin{cases} I_1, & |S| > |S_{gr}|, \\ I_2, & |S| \le |S_{gr}|. \end{cases} \tag{5.987}$$

As noted earlier, the functional I_2 is minimized by control that is not limited in modulus, therefore, for control (5.178) the functional I_2 will have the form (5.175).

To define the functional I_1, we introduce the notation

$$S' = S + \sqrt{|S|}\,\text{sign}[S], \tag{5.988}$$

when taking it into account, algorithm (5.178) can be represented as follows

$$U = -\begin{cases} \text{sign}\left(S'\right), & |S'| > 1, \\ S', & |S'| \le 1. \end{cases} \tag{5.989}$$

The form of the first term in the control (5.181) is similar to the control (5.133) that minimizes the functional (5.131).

Thus, taking into account the adopted designation (5.180) and the quality functional (5.175) defined for the control system without taking into account the constraint on the control signal, the functional that is minimized by the control (5.178) will take the form

$$I=\frac{1}{2C}\int_0^\infty\left[\begin{cases}|S+\sqrt{|S|}\,\mathrm{sign}[S]|+1, & |S|>|S_{gr}|,\\ \left(\frac{2\sqrt{|S|^3}}{3}+\frac{1}{2}|S|^2\right)+\left(\frac{|U|^3}{3}+|U|^2\right)dt, & |S|\leq|S_{gr}|.\end{cases}\right]dt. \tag{5.990}$$

5.2.7. Construction of quality functionals for control systems with exponential activation function

Control systems with an irrational activation function, being essentially non-linear, occupy an intermediate position between linear and relay systems. The controls implemented in such systems are limiting cases of controls with irrational activation functions, confirming the hypothesis that systems with irrational activation functions are generalizations of known linear and relay systems.

All of the above creates the prerequisites for a controlled modification of the exponent of the activation function α of the control input (5.152). In this case, the optimal control will be sought in the class of functions of the form

$$U=-f_1(S)^{f_2(S)}. \tag{5.991}$$

Before proceeding to the definition of the control objective, the achievement of which is carried out by control (5.183), we define the quality functional, the minimization of which is carried out by control of the type

$$\begin{aligned}&U=-C^{|f_2(S)|}\,\mathrm{sign}[f_2(S)],\\&C\neq|1|,\\&C\neq 0.\end{aligned} \tag{5.992}$$

Strictly speaking, unlike the control (5.183), control (5.184) is discontinuous and, depending on the constant C, implements two different static characteristics, shown in Fig. 5.19–5.20.

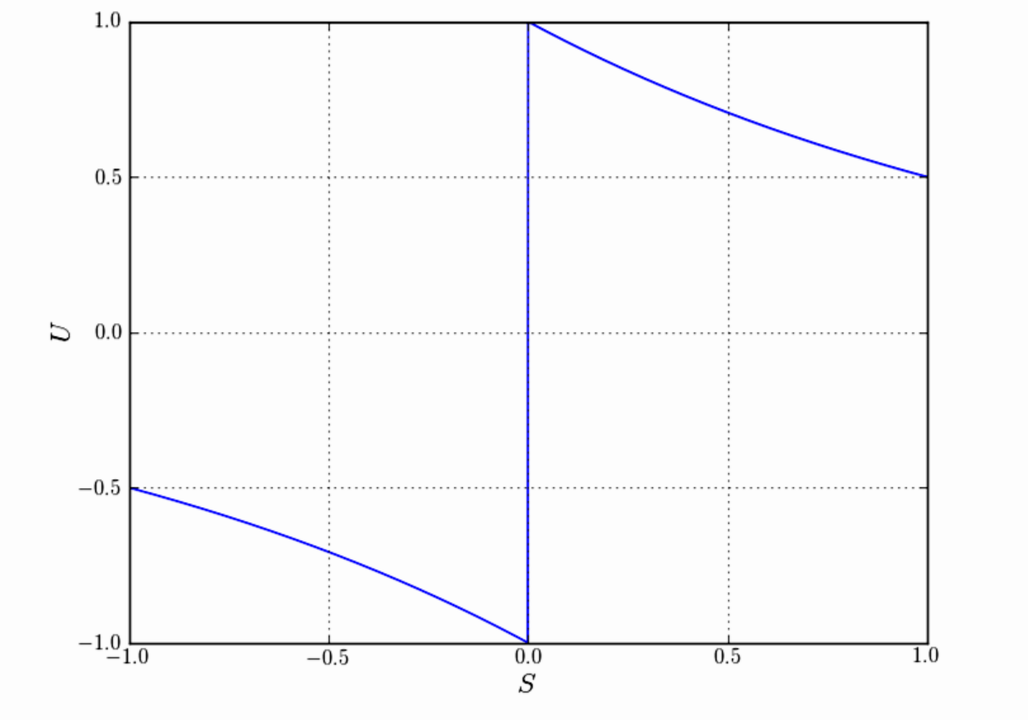

Fig. 5.19. Static characteristics of the optimal controller with exponential activation function, $U = f(S)$ at $C = 0,5$

The presence of first-order discontinuities in control (5.184) does not ensure the formation of high-order sliding modes. However, the consideration of this control is of a methodological nature, associated with the clarification of the foundations of the synthesis of optimal controls with an exponential activation function.

As before, we will look for the minimized quality functional in the form (5.84).

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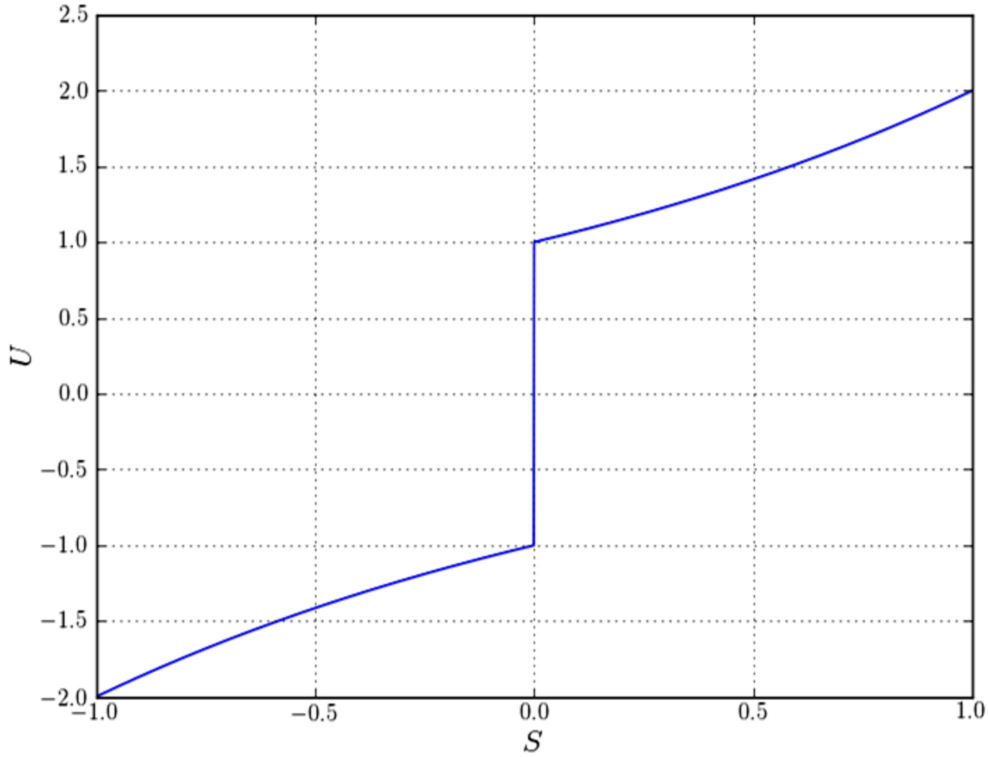

Fig. 5.20. Static characteristics of the optimal controller with exponential activation function, $U = f(S)$ at $C = 2$

In accordance with the expressions (5.103) for control (5.184), we determine the conjugate functions $f(S)$ and $g(U)$

$$f(S) = -g^{-1}(U) = C^{|f_2(S)|}\operatorname{sign}[f_2(S)],$$
$$g(U) = -f^{-1}(S) = \log_C |U| \operatorname{sign}[U]. \qquad (5.993)$$

The graphs of the inverse functions $g(U)$ corresponding to the functions shown in Fig. 5.19–5.20 are shown in Fig. 5.21–5.22.

Then, integrating the second expression of the system (5.185) with respect to the control input allows us to uniquely determine the component $G(U)$ that determines the control energy consumption

$$G(U) = \int g(U)dU = \int \log_C |U| \operatorname{sign}[U] dU = \frac{1}{\ln C}\int \ln|U| \operatorname{sign}[U] dU =$$
$$= \frac{|U|\left(\ln|U|-1\right)}{\ln C} = C_1 |U|\left(\ln|U|-1\right) + \text{Const},\ C_1 = \ln C. \qquad (5.994)$$

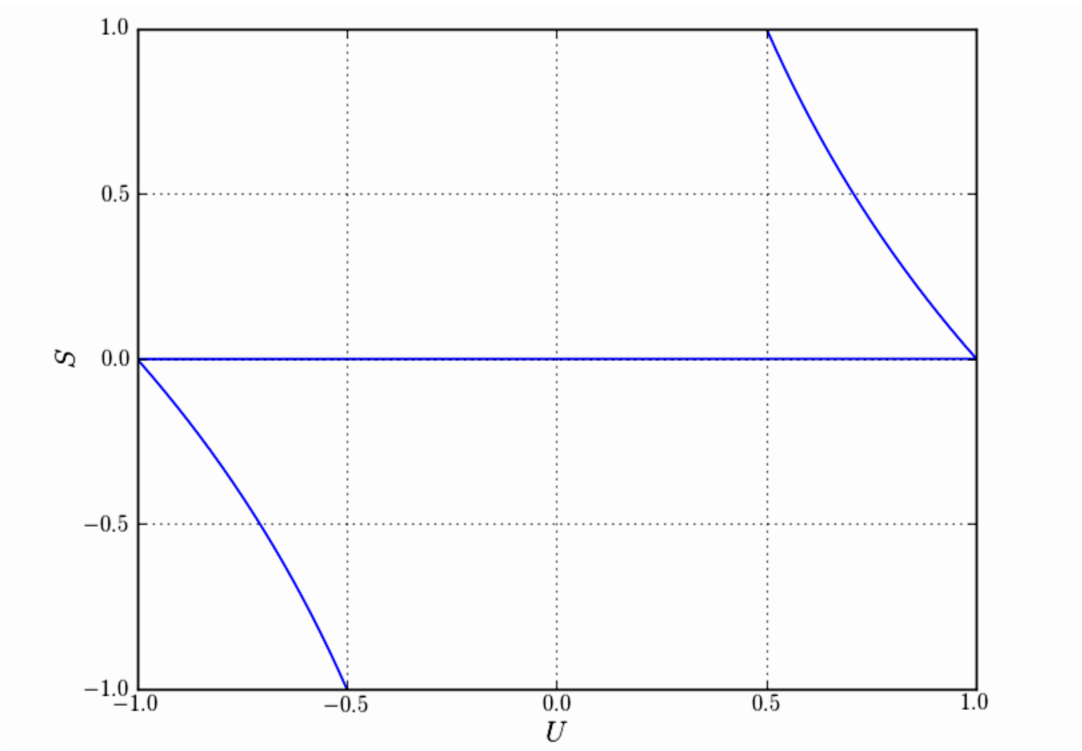

Fig. 5.21. Inverse static characteristics of the optimal controller with exponential activation function, $S = g(U)$ at $C = 0,5$

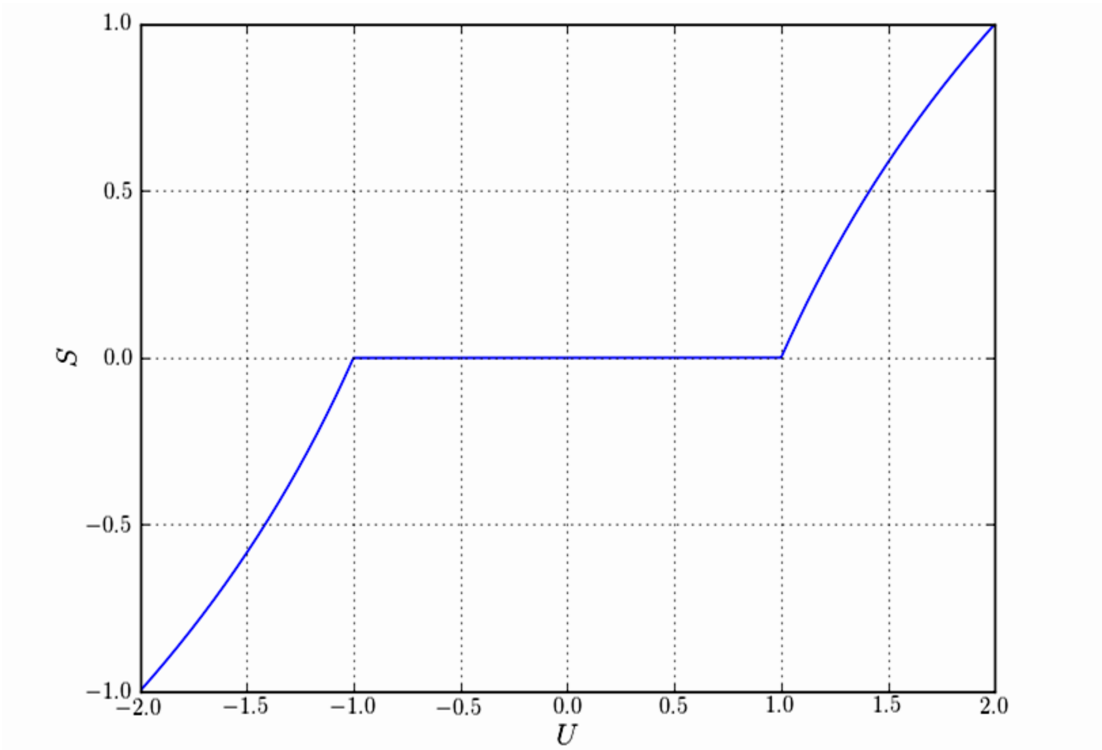

Fig. 5.22. Inverse static characteristics of the optimal controller with exponential activation function, $S = g(U)$ at $C = 2$

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As follows from the analysis of the expression (5.186), the function $G(U)$ is sign-constant and has a clearly defined extremum.

To find the component $F(\eta_0,\eta_1,\eta_2,\eta_3)$ that determines the motion of the control system, we write the following expression

$$F(\eta_0,\eta_1,\eta_2,\eta_3)=Sf(S)-\int S\frac{\partial f(S)}{\partial S}dS=$$
$$=SC^{|f_2(S)|}\operatorname{sign}[f_2(S)]-\int S\frac{\partial C^{f_2(S)}}{\partial f_2(S)}\frac{\partial f_2(S)}{\partial S}dS=$$
$$=SC^{|f_2(S)|}\operatorname{sign}[f_2(S)]-\ln C\int SC^{|f_2(S)|}\operatorname{sign}[f_2(S)]\frac{\partial f_2(S)}{\partial S}dS. \quad (5.995)$$

Determining the integral forming the second term of expression (5.187) in general form for an indefinite function $f_2(S)$ is difficult. Therefore, it makes sense to consider finding the value of the expression (5.187) for different functions $f_2(S)$.

We will begin the determination of the function (5.187) with the case

$$f_2(S)=S \quad (5.996)$$

Substituting the value of the function (5.188) into the expression (5.187), we obtain

$$F(\eta_0,\eta_1,\eta_2,\eta_3)=SC^{|S|}\operatorname{sign}[S]-\ln C\int SC^{|S|}\operatorname{sign}[S]\frac{\partial S}{\partial S}dS=$$
$$=SC^{|S|}\operatorname{sign}[S]-\ln C\int SC^{|S|}\operatorname{sign}[S]dS=$$
$$=|S|C^{|S|}-\frac{\ln C|S|C^{|S|}}{\ln C}+\frac{C^{|S|}}{\ln C}=\frac{C^{|S|}}{\ln C}=\frac{C^{|S|}}{C_1}. \quad (5.997)$$

Substituting the values of the expressions (5.189) and (5.186) into the integral (5.84), we obtain the following quality functional, which is minimized by the control (5.184)

$$I=\int_0^\infty\left[\left(\frac{C^{|S|}}{C_1}\right)+\left(C_1|U|\ln|U|-C_1|U|\right)\right]dt. \quad (5.998)$$

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As follows from the analysis of the functional (5.190), to implement controls of the type (5.184), the minimized functional must contain the logarithm of the control input and the exponential function S.

Let us take as a function $f_2(S)$ defining the component (5.187) of the functional (5.84) an irrational function

$$f_2(S) = \sqrt{|S|}. \tag{5.999}$$

Substituting the value of the function (5.191) into the expression (5.187), we obtain

$$F(\eta_0, \eta_1, \eta_2, \eta_3) = 2\frac{C^{\sqrt{|S|}}}{C_1}\left(\sqrt{|S|} - \frac{1}{C_1}\right). \tag{5.1000}$$

Substituting the values of the expressions (5.192) and (5.186) into the integral (5.84), we obtain the quality functional, which is minimized by the control (5.184)

$$I = \int_0^{\infty}\left[2\frac{C^{\sqrt{|S|}}}{C_1}\left(\sqrt{|S|} - \frac{1}{C_1}\right) + \left(C_1|U|\ln|U| - C_1|U|\right)\right]dt. \tag{5.1001}$$

Using a hyperbolic function as a function $f_2(S)$ that determines a component (5.187) of a functional (5.84)

$$f_2(S) = \frac{1}{|S|} \tag{5.1002}$$

allows us to obtain the expression

$$F(\eta_0, \eta_1, \eta_2, \eta_3) = |S|C^{1/|S|} + \ln C\int\frac{C^{1/|S|}}{|S|}dS. \tag{5.1003}$$

The integral that forms the second term of the expression (5.195)cannot be defined using elementary functions. However, using the exponential integral

$$Ei(a, z) = \int_1^{\infty} e^{-k_1 z} k_1^{-a} dk_1 \tag{5.1004}$$

allows us to represent the expression (5.195) in the form

$$F(\eta_0,\eta_1,\eta_2,\eta_3)=|S|C^{1/|S|}+\ln C Ei\left(1,-\frac{lnC}{|S|}\right). \tag{5.1005}$$

Substituting the values of the expressions (5.197) and (5.186) into the integral (5.84), we obtain the quality functional, which is minimized by the control (5.184)

$$I=\int_0^\infty\left[\left(|S|C^{1/|S|}+\ln C Ei\left(1,-\frac{\ln C}{|S|}\right)\right)+C_1|U|\left(\ln|U|-1\right)\right]dt \quad .(5.1006)$$

Using the upper incomplete gamma function expression to calculate the second term (5.187)

$$\Gamma(s,x)=\int_x^\infty t^{s-1}e^{-t}dt, \tag{5.1007}$$

which is related to the integral (5.196) by the relation

$$Ei(a,z)=z^{a-1}\Gamma(1-a,z) \tag{5.1008}$$

allows us to generalize the obtained results to the case of an arbitrary exponent α in the function $f_2(S)$.

As an example, we define a quality functional that is minimized by optimal control

$$U=-C^{|S|^\alpha}\operatorname{sign}(S),\alpha\neq 0. \tag{5.1009}$$

For control (5.201), the expression (5.187) will take the form

$$F(\eta_0,\eta_1,\eta_2,\eta_3)=|S|C^{|S|^\alpha}+\frac{\Gamma(\frac{\alpha+1}{\alpha},-|S|^\alpha\ln C)}{\sqrt[\alpha]{(-1)\alpha\ln C}}. \tag{5.1010}$$

Substituting the expressions (5.202) and (5.186) into the integral (5.84), we obtain the quality functional, which is minimized by the control (5.201)

$$I=\int_0^\infty\left[\left(|S|C^{|S|^\alpha}+\frac{\Gamma(\frac{\alpha+1}{\alpha},-|S|^\alpha\ln C)}{\sqrt[\alpha]{(-1)\alpha\ln C}}\right)+C_1|U|\left(\ln|U|-1\right)\right]dt. \tag{5.1011}$$

Attempts to further complicate the function $f_2(S)$, in particular, transitioning to additive forms of the type

$$f_2(S) = \sum_{i=1}^{m} w_i f_i(S) \tag{5.1012}$$

lead to the necessity of using non-elementary functions to determine the value of the integral

$$I = \int S \frac{\partial C^{f_2(S)}}{\partial f_2(S)} \frac{\partial f_2(S)}{\partial S} dS. \tag{5.1013}$$

In this case, it is possible not only to expand the range of solutions of the integral (5.205), but also, as a consequence, to expand the class of functionals (5.84), and therefore to increase the domain of possible controls.

Thus, the use of the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{5.1014}$$

allows us for a function $f_2(S)$ of the form

$$f_2(S) = w_1 \sqrt{|S|} + w_2 |S| \tag{5.1015}$$

to find the value of the integral (5.205) minimized by the control input

$$U = -C^{w_1 \sqrt{|S|} + w_2 |S|} \operatorname{sign}[S]. \tag{5.1016}$$

However, due to the complexity of the resulting integrand, it is not considered here.

The calculations given in the previous sections show that for optimal controls with a wide class of activation functions, a minimized integral quality functional can be found analytically. Moreover, in general, this functional may not be defined through elementary mathematical functions.

Let us define the quality functional, the minimization of which is carried out by the optimal control of the form (5.183). Such control is quite general in terms of realizing different static characteristics.

According to the system (5.103), the conjugate expression for the dependence (5.183) is

$$g(U) = -S = -f_1^{-1}\left(\frac{\ln U}{\ln f_2(S)}\right). \tag{5.1017}$$

Accordingly, the component $G(U)$ of the functional (5.84) is determined by the integral

$$G(U) = \int g(U)dU = -\int f_2^{-1}\left(\frac{\ln U}{\ln f_1(S)}\right)dU. \tag{5.1018}$$

By differentiating the equation (5.183), we find the total derivative

$$\frac{dU}{dt} = \left(\frac{\partial f_1(S)^{f_2(S)}}{\partial f_1(S)}\frac{\partial f_1(S)}{\partial S} + \frac{\partial f_1(S)^{f_2(S)}}{\partial f_2(S)}\frac{\partial f_2(S)}{\partial S}\right)\frac{dS}{dt}. \tag{5.1019}$$

In accordance with the derivative (5.211), the differential dU is determined by the expression

$$dU = \left(\frac{\partial f_1(S)^{f_2(S)}}{\partial f_1(S)}\frac{\partial f_1(S)}{\partial S} + \frac{\partial f_1(S)^{f_2(S)}}{\partial f_2(S)}\frac{\partial f_2(S)}{\partial S}\right)dS. \tag{5.1020}$$

Substituting the differential (5.212) into the integral (5.210) allows us to determine the component $G(U)$ of the functional (5.84) that is minimized by the control (5.183)

$$G(U) = \int S\left(\frac{\partial f_1(S)^{f_2(S)}}{\partial f_1(S)}\frac{\partial f_1(S)}{\partial S} + \frac{\partial f_1(S)^{f_2(S)}}{\partial f_2(S)}\frac{\partial f_2(S)}{\partial S}\right)dS. \tag{5.1021}$$

Using expressions (5.109) and (5.213) allows us to define the component $F(\eta_1, \cdots, \eta_n)$ of the functional (5.84)

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$$F(\eta_1,\cdots,\eta_n) = Sf_1(S)^{f_2(S)} -$$
$$-\int S\left(\frac{\partial f_1(S)^{f_2(S)}}{\partial f_1(S)}\frac{\partial f_1(S)}{\partial S} + \frac{\partial f_1(S)^{f_2(S)}}{\partial f_2(S)}\frac{\partial f_2(S)}{\partial S}\right)dS. \quad (5.1022)$$

A generalization of the expression (5.183) is a control of the form

$$U = -f_1(S_1)^{f_2(S_2)}, \quad (5.1023)$$

where

$$S_1 = s_1(\eta_i),\ S_2 = s_2(\eta_i). \quad (5.1024)$$

By analogy with expressions (5.210) and (5.214), functions $F(\eta_1,\cdots,\eta_n)$ and $G(U)$ for such control are

$$G(U) = \int g(U)dU = -\int f_2^{-1}\left(\frac{\ln U}{\ln f_1(S_1)}\right)dU \quad (5.1025)$$

and

$$F(\eta_1,\cdots,\eta_n) = S_2 f_1(S_1)^{f_2(S_2)} -$$
$$-\int S_2\left(\frac{\partial f_1(S_1)^{f_2(S_2)}}{\partial f_1(S_1)}\frac{\partial f_1(S_1)}{\partial S_1} + \frac{\partial f_1(S_1)^{f_2(S_2)}}{\partial f_2(S_2)}\frac{\partial f_2(S_2)}{\partial S_2}\right)dS_2$$

(5.1026)

respectively.

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CONCLUSIONS

The research results presented in the text allow us to formulate the following conclusions:

• Feedback linearization at the stage of mathematical description of the control object allows arbitrary differential equations to be brought to the standard Brunovsky form and creates the prerequisites for the synthesis of an abstract controller, whose operation algorithm does not depend on the parameters and structure of the object.

• Regularization of the equations describing the dynamics of the control object allows one to set the desired distribution of the roots of the characteristic equation , thereby ensuring the required quality of the control process.

• Setting the desired trajectories of the roots of the characteristic equation expands the applicability of modal controls and allows one to synthesize both continuous and discontinuous systems.

• Depending on the selected trajectories of movement of the roots of the characteristic equation, high-order sliding modes may arise in a closed-loop system. Control systems in which high-order sliding modes occur are characterized by reduced energy consumption compared to classical discontinuous systems.

• One of the methods for implementing higher-order sliding modes is to use irrational activation functions in control algorithms. The use of these functions leads to the creation of nonlinear regulators that ensure transition process quality comparable to that in relay systems.

• Control algorithms with an irrational activation function can be classified as super-twisting algorithms, which provide high dynamic and static performance of closed-loop systems.

• Transition functions of closed-loop systems with irrational activation functions can be determined analytically by solving dynamic equations, which, from a mathematical point of view, are Abel equations of the second kind. The obtained analytical dependencies make it possible to determine the control time in closed-loop systems and to construct the trajectories of their movement.

• The control algorithms obtained by root methods formed the basis of the inverse dynamic programming problem, the solution of which made it possible to determine quality functionals, the minimization of which is carried out under the condition of the existence of high-order sliding modes.

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