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ON CHAOTIC ATTRACTORS WHOSE BASINS OF ATTRACTION COINCIDE WITH THE WHOLE SPACE

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Abstract. A new type of chaotic attractors, whose basin of attraction is the entire phase space, is considered. The main difference between these attractors and the known ones is that any trajectory starting from the basin of attraction first enters a unique transport channel (which is a straight line), and then the trajectory reaches the attractor itself along this channel. For any quadratic dynamic system generating the mentioned attractor, a new concept of a uniquely defined degenerate autonomous quadratic dynamic system with exactly one real double equilibrium point is introduced. It is shown that if the degenerate system exhibits chaotic behavior, then the original (non-degenerate) system also exhibits similar chaotic behavior.

Key words: complexification and realification of dynamical systems, projection on subspace, degenerate system, chaotic attractor.

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1. Introduction

Chaos is a very interesting nonlinear phenomenon, which has been intensively studied in the last five decades. Many potential applications have come true in engineering, medicine, laser and biological systems, and other areas (see, for example, [2–11, 13, 14, 17–19] and the many references cited therein).

In the last twenty years, among numerous research directions of one or another question of the theoretical nonlinear dynamics, it is possible especially to select one of such directions containing so-called dynamic systems without equilibrium points. The behavior of such systems was studied in papers [4, 14, 18, 19].

Chaotic systems without equilibrium points have applications in the field of secure communication because in such systems there are no limitations on the number of equilibrium points. In addition to this property (absence of equilibrium points), chaotic systems used in secure communication problems must also have the property that their solutions must be bounded for any initial data (this is

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desirable). In this case, the key to decrypt the encoded messages becomes more complex (see [12, 13]).

In this regard, the aim of this work is to find out whether there exist some systems of differential equations whose attractor structure would be different from the structure of attractors known at present.

2. Complexification and Realification of Dynamical Systems

Denote by $\mathbb{R}^n$ a real space of dimension n. Let $\mathbf{x} = (x_1, ..., x_n)^T \in \mathbb{R}^n$ be an unknown vector whose coordinates are functions of time t. Let also $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^n$ be a real vector function of the variable $\mathbf{x}$.

Consider the autonomous real dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) \equiv (f_1(\mathbf{x}), ..., f_n(\mathbf{x}))^T, \tag{2.1}$$

where functions $f_1(\mathbf{x}), ..., f_n(\mathbf{x})$ are real polynomials.

In what follows, we will assume that the functions $f_1(\mathbf{x}), ..., f_n(\mathbf{x})$ are algebraically independent. This means that there does not exist nonzero real polynomial $\Phi(\xi_1, ..., \xi_n)$ depending on n variables $\xi_1, ..., \xi_n$ such that $\forall \mathbf{x} \in \mathbb{R}^n$ $\Phi(f_1(\mathbf{x}), ..., f_n(\mathbf{x})) \equiv 0$.

Introduce a complex n-dimensional vector $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, where $i = \sqrt{-1}$, and $\mathbf{x}, \mathbf{y}$ are real n-dimensional vectors.

Change the variable $\mathbf{x} \to \mathbf{z}$ in the system (2.1). Then from (2.1) it follows that $\dot{\mathbf{z}} = \dot{\mathbf{x}} + i\dot{\mathbf{y}} = \mathbf{f}(\mathbf{z}) = \mathbf{u}(\mathbf{x}, \mathbf{y}) + i\mathbf{v}(\mathbf{x}, \mathbf{y})$, where $\mathbf{u}(\mathbf{x}, \mathbf{y}), \mathbf{v}(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n$ are known real vector functions. (In addition, we have $\mathbf{u}(\mathbf{x}, \mathbf{0}) = \mathbf{f}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x}, \mathbf{0}) = \mathbf{0}$.)

Definition 2.1. The system

$$\dot{\mathbf{z}}(t) = \mathbf{f}(\mathbf{z}), \mathbf{z} \in \mathbb{C}^n, \mathbf{f}(\mathbf{z}) \in \mathbb{C}^n \tag{2.2}$$

is called a complexification of the system (2.1), and the system

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{y}}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{u}(\mathbf{x}, \mathbf{y}) \\ \mathbf{v}(\mathbf{x}, \mathbf{y}) \end{pmatrix}, \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{2n}, \begin{pmatrix} \mathbf{u}(\mathbf{x}, \mathbf{y}) \\ \mathbf{v}(\mathbf{x}, \mathbf{y}) \end{pmatrix} \in \mathbb{R}^{2n} \tag{2.3}$$

is called a realification of the system (2.2).

Let $\mathbb{W} \subset \mathbb{C}^n$ be an algebraic variety of all complex solutions of system $\mathbf{f}(\mathbf{z}) = \mathbf{0}$ [16]. (By virtue of the definition of function $\mathbf{f}(\mathbf{z})$ the variety $\mathbb{W}$ is a finite nonzero set of points. The number of these points is called a degree of the variety $\mathbb{W}$ [16].)

Denote by $\deg_{\mathbb{C}}\mathbb{W}$ the number of points in the variety $\mathbb{W}$.

Definition 2.2. The system (2.2) is called complete if $\deg_{\mathbb{C}}\mathbb{W} = \deg_{\mathbb{C}}\mathbb{L}$, where $\mathbb{L} \subset \mathbb{C}^n$ is the variety of all solutions of system (2.2) for which all its coefficients are independent parameters.

The elimination theory asserts that the system $\mathbf{f}(\mathbf{z}) = \mathbf{0}$ with the help of algebraic transformations can always be reduced to one polynomial equation $\Delta(s) = a_0 s^l + a_1 s^{l-1} + ... + a_l = 0$ [16], where $l = \deg_{\mathbb{C}} \mathbb{W}$; $s = s(z_1, ..., z_n)$ is a linear function of variables $z_1, ..., z_n$. (Note that the numbers $a_0, ..., a_l$ are real.)

Assume that the real polynomial $\Delta(s)$ has $l_{\mathbb{R}}$ real and $l_{\mathbb{C}}$ complex roots (any complex root has the conjugate root). Thus, $l = \deg_{\mathbb{C}} \mathbb{W} = l_{\mathbb{R}} + l_{\mathbb{C}}$ and $l_{\mathbb{C}}$ is an even number.

Definition 2.3. The ordered pair of integer positive numbers $(l_{\mathbb{R}}, l_{\mathbb{C}})$ is called a signature of the system (2.2) (or system (2.1)).

Let $(c_1, ..., c_n)^T$ and $A = (a_{ij}), B_1, ..., B_n \in \mathbb{R}^{n \times n}$ be a real vector and real matrices, and let the matrices $B_1, ..., B_n$ be symmetrical.

Consider the system of ordinary quadratic differential equations

$$\begin{cases} \dot{x}_1(t) = \sum_{j=1}^{n} a_{1j} x_j(t) + \mathbf{x}^T(t) B_1 \mathbf{x}(t) + c_1 \equiv f_1(\mathbf{x}(t)), \\ \cdots\cdots\cdots\cdots\cdots\cdots, \\ \dot{x}_n(t) = \sum_{j=1}^{n} a_{nj} x_j(t) + \mathbf{x}^T(t) B_n \mathbf{x}(t) + c_n \equiv f_n(\mathbf{x}(t)) \end{cases} \tag{2.4}$$

of order n with the vector of initial values $\mathbf{x}^T(0) = (x_{10}, ..., x_{n0})$.

Consider the complexification

$$\begin{cases} \dot{z}_1(t) = \sum_{j=1}^{n} a_{1j} z_j(t) + \mathbf{z}^T(t) B_1 \mathbf{z}(t) + c_1 \equiv f_1(\mathbf{z}(t)), \\ \cdots\cdots\cdots\cdots\cdots\cdots, \\ \dot{z}_n(t) = \sum_{j=1}^{n} a_{nj} z_j(t) + \mathbf{z}^T(t) B_n \mathbf{z}(t) + c_n \equiv f_n(\mathbf{z}(t)) \end{cases} \tag{2.5}$$

of system (2.4). The realification of system (2.5) is the following $2n$-dimensional real system:

$$\begin{cases} \dot{x}_1(t) = \sum_{j=1}^{n} a_{1j} x_j(t) + \mathbf{x}^T(t) B_1 \mathbf{x}(t) - \mathbf{y}^T(t) B_1 \mathbf{y}(t) + c_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots, \\ \dot{x}_n(t) = \sum_{j=1}^{n} a_{nj} x_j(t) + \mathbf{x}^T(t) B_n \mathbf{x}(t) - \mathbf{y}^T(t) B_n \mathbf{y}(t) + c_n, \\ \dot{y}_1(t) = \sum_{j=1}^{n} a_{1j} y_j(t) + \mathbf{y}^T(t) B_1 \mathbf{x}(t) + \mathbf{x}^T(t) B_1 \mathbf{y}(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots, \\ \dot{y}_n(t) = \sum_{j=1}^{n} a_{nj} y_j(t) + \mathbf{y}^T(t) B_n \mathbf{x}(t) + \mathbf{x}^T(t) B_n \mathbf{y}(t). \end{cases} \tag{2.6}$$

Compute the Jacobi matrix of system (2.6). Then we have

$$J(\mathbf{x},\mathbf{y}) = \begin{pmatrix} A + B(\mathbf{x}), & -B(\mathbf{y}) \\ B(\mathbf{y}), & A + B(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{2n\times 2n},$$

where the matrices

$$B(\mathbf{x}) = \Big((\underbrace{0,...,0,j}_{j},0,...,0)B_k\mathbf{x} + \mathbf{x}^T B_k(\underbrace{0,...,0,j}_{j},0,...,0)^T\Big)_{j,k=1,...,n} \in \mathbb{R}^{n\times n}$$

and the matrices

$$B(\mathbf{y}) = \Big((\underbrace{0,...,0,j}_{j},0,...,0)B_k\mathbf{y} + \mathbf{y}^T B_k(\underbrace{0,...,0,j}_{j},0,...,0)^T\Big)_{j,k=1,...,n} \in \mathbb{R}^{n\times n}$$

are symmetric.

We make use of the following result.

Theorem 2.1. *[1] Let* $\mathbf{g}$ *and* $\mathbf{h}$ *be real* n*-dimensional vectors. Then the eigenvalues of the matrix* $J(\mathbf{g},\mathbf{h})$ *are the eigenvalues of the matrices* $A + B(\mathbf{g}) + iB(\mathbf{h})$ *and* $A^T + B^T(\mathbf{g}) - iB^T(\mathbf{h})$.

Lemma 2.1. *Let* $\begin{pmatrix} \mathbf{g}^* \\ \mathbf{h}^* \end{pmatrix} \in \mathbb{R}^{2n}$ *be an equilibrium point of system (2.6);* $\mathbf{h}^* \neq \mathbf{0}$. *Then* $\begin{pmatrix} \mathbf{g}^* \\ -\mathbf{h}^* \end{pmatrix} \in \mathbb{R}^{2n}$ *is also an equilibrium point of system (2.6) and, therefore, there exists an even number of eigenvectors of the matrix* $J(\mathbf{g},\mathbf{h})$ *such that* $\mathbf{h}^* \neq \mathbf{0}$.

Proof. The statement of Lemma 2.1 follows from the form of system (2.6).

Lemma 2.2. *The equilibrium points* $\begin{pmatrix} \mathbf{g}^* \\ \mathbf{h}^* \end{pmatrix} \in \mathbb{R}^{2n}$ *and* $\begin{pmatrix} \mathbf{g}^* \\ -\mathbf{h}^* \end{pmatrix} \in \mathbb{R}^{2n}$ *have the same type.*

Proof. Consider the Jacobi matrices in these points:

$$J(\mathbf{g}^*,\mathbf{h}^*) = \begin{pmatrix} A + B(\mathbf{g}^*), & -B(\mathbf{h}^*) \\ B(\mathbf{h}^*), & A + B(\mathbf{g}^*) \end{pmatrix},$$

$$J(\mathbf{g}^*,-\mathbf{h}^*) = \begin{pmatrix} A + B(\mathbf{g}^*), & B(\mathbf{h}^*) \\ -B(\mathbf{h}^*), & A + B(\mathbf{g}^*) \end{pmatrix}.$$

From Theorem 2.1 it follows that the eigenvalues of the matrix $J(\mathbf{g}^*,\mathbf{h}^*)$ are eigenvalues of the matrices

$$A + B(\mathbf{g}^*) + iB(\mathbf{h}^*), A^T + B^T(\mathbf{g}^*) - iB^T(\mathbf{h}^*), \tag{2.7}$$

and the eigenvalues of the matrix $J(\mathbf{g}^*,-\mathbf{h}^*)$ are eigenvalues of the matrices

$$A + B(\mathbf{g}^*) - iB(\mathbf{h}^*), A^T + B^T(\mathbf{g}^*) + iB^T(\mathbf{h}^*). \tag{2.8}$$

The eigenvalues of the matrices $A+B(\mathbf{g}^*)+iB(\mathbf{h}^*)$ and $A^T+B^T(\mathbf{g}^*)+iB^T(\mathbf{h}^*)$ ($A+B(\mathbf{g}^*)-iB(\mathbf{h}^*)$ and $A^T+B^T(\mathbf{g}^*)-iB^T(\mathbf{h}^*)$) are coincide. Then it is clear that eigenvalues of matrices (2.7) coincide with the eigenvalues of matrices (2.8). The proof is finished. □

Let $\mathbf{Pr}:\mathbb{R}^{2n}\to\mathbb{R}^n$ be a mapping of projection from $\mathbb{R}^{2n}$ onto $\mathbb{R}^n$ is operating by the rule: $\forall(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2n}$ $\mathbf{Pr}(\mathbf{x},\mathbf{y})=\mathbf{x}$. Here $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ and $\mathbb{R}^{2n}=\mathbb{R}^n\times\mathbb{R}^n$ is the Cartesian product.

Introduce in system (2.6) the new variable $(\mathbf{x}_{new},\mathbf{y}_{new})^T$ by the formula

$$\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\to\begin{pmatrix}\mathbf{x}_{new}+\mathbf{g}^*\\ \mathbf{y}_{new}+\mathbf{h}^*\end{pmatrix}.$$

Then we obtain

$$\begin{pmatrix}\dot{\mathbf{x}}(t)\\ \dot{\mathbf{y}}(t)\end{pmatrix}$$

$$=\begin{pmatrix}A+B(\mathbf{g}^*), & -B(\mathbf{h}^*)\\ B(\mathbf{h}^*), & A+B(\mathbf{g}^*)\end{pmatrix}\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{y}(t)\end{pmatrix}+\begin{pmatrix}\mathbf{x}^T(t)B_1\mathbf{x}(t)-\mathbf{y}^T(t)B_1\mathbf{y}(t)\\ \cdots\\ \mathbf{x}^T(t)B_n\mathbf{x}(t)-\mathbf{y}^T(t)B_n\mathbf{y}(t)\\ \mathbf{y}^T(t)B_1\mathbf{x}(t)+\mathbf{x}^T(t)B_1\mathbf{y}(t)\\ \cdots\\ \mathbf{y}^T(t)B_n\mathbf{x}(t)+\mathbf{x}^T(t)B_n\mathbf{y}(t)\end{pmatrix}. \tag{2.9}$$

It is clear that the point $(\mathbf{0},\mathbf{0})^T\in\mathbb{R}^{2n}$ is the equilibrium point of the system (2.9). (For simplicity we have left the former designations of variables $\mathbf{x}$ and $\mathbf{y}$, and corresponding coefficients.)

It is quite obvious that if system (2.9) is chaotic then the system (2.6) (or (2.4)) will be the same (and vice versa). However, the system (2.9) is $2n$-dimensional system. Therefore, we do an attempt to simplify this situation and if it is possible, then we will reduce the problem of existence of chaos in $2n$-dimensional systems to the same problem for some n-dimensional systems.

Consider the projection $\mathbf{Pr}(\mathbf{x},\mathbf{y})=\mathbf{x}$ of solutions $(\mathbf{x}_{new}(t),\mathbf{y}_{new}(t))^T\in\mathbb{R}^{2n}$ of system (2.9) on subspace $\mathbb{R}^n$. If $\mathbf{h}^*=\mathbf{0}$, then the solution that we have again $\mathbf{x}_{new}(t)-\mathbf{g}^*\in\mathbb{R}^n$ will satisfy system (2.4). On the other hand, if $\mathbf{h}^*\neq\mathbf{0}$, we introduce the system

$$\dot{\mathbf{x}}_d(t)=(A+B(\mathbf{g}^*))\mathbf{x}_d(t)+\begin{pmatrix}\mathbf{x}_d^T(t)B_1\mathbf{x}_d(t)\\ \cdots\\ \mathbf{x}_d^T(t)B_n\mathbf{x}_d(t)\end{pmatrix}. \tag{2.10}$$

Thus, from (2.6) it follows that if $\mathbf{g}^*\neq\mathbf{0}$ and $\mathbf{h}^*\neq\mathbf{0}$, then $[A+B(\mathbf{g}^*)]\mathbf{h}^*=\mathbf{0}$ and, therefore, $\det[A+B(\mathbf{g}^*)]=\det\mathbf{J}(\mathbf{g}^*)=0$. If $\mathbf{g}^*=\mathbf{0}$ and $\mathbf{h}^*\neq\mathbf{0}$, then it must be $\det A=\det\mathbf{J}(\mathbf{0})=0$. It allows us to introduce the following definition.

Definition 2.4. If $\mathbf{h}^*\neq\mathbf{0}$, then the system (2.10) is called degenerate.

Comment. Note that if $\mathbf{h}^* = \mathbf{0}$, then the degenerate system (2.10) is the system that can be obtained from the system (2.4) by moving the origin from the point $\mathbf{0}$ to the point $\mathbf{g}^*$. In this case, the concept of the degenerate system is trivial.

Suppose that system (2.10) has the signature $(0, l_{\mathbb{C}})$. Then there exist $m = l_{\mathbb{C}}/2$ pairs of the equilibrium points of system (2.6) (see Lemma 2.1):

$$\begin{pmatrix} \mathbf{g}^*{}_1 \\ \mathbf{h}^*{}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{g}^*{}_1 \\ -\mathbf{h}^*{}_1 \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{g}^*{}_m \\ \mathbf{h}^*{}_m \end{pmatrix}, \begin{pmatrix} \mathbf{g}^*{}_m \\ -\mathbf{h}^*{}_m \end{pmatrix}.$$

Thus, for any system (2.4) we can construct m degenerate systems. In addition, any degenerate system has one real equilibrium point of multiple 2 and this equilibrium point is the point $\mathbf{0} \in \mathbb{R}^n$.

Notice that for system (2.10) the polynomial $\Delta(s) = a_0 s^l + a_1 s^{l-1} + \ldots + a_l = 0$ has only one real double root. Thus, for system (2.10) we have $l = l_{\mathbb{C}}$ and the signature of system (2.10) is $(2, l_{\mathbb{C}} - 2)$.

Lemma 2.3. *Let the point $H(\mathbf{g}^*, \mathbf{h}^*)$ be the equilibrium point of the system (2.6) and let the norm $\|\mathbf{h}^*\|$ be small enough. If system (2.10) constructed for appropriate vector $\mathbf{g}^*$ is chaotic, then system (2.6) (and system (2.4)) will also be chaotic.*

Proof. We take advantage of the following known fact of algebraic geometry [16].

Let $\mathbb{W} \subset \mathbb{R}^{2n}$ be a linear d-dimensional subspace in $\mathbb{R}^{2n}$ and let $\mathbb{A} \subset \mathbb{R}^{2n} - \mathbb{W} \subset \mathbb{R}^{2n}$ be a chaotic attractor in the subset $\mathbb{R}^{2n} - \mathbb{W}$. (Here we consider the case $2n - d - 1 < \dim \mathbb{A} < 2n - d$ and $\mathbb{A} \cap \mathbb{W} = 0$.) Let also $\mathbf{Pr}_{\mathbb{W}}(\mathbb{A}) : \mathbb{R}^{2n} - \mathbb{W} \to \mathbb{W}$ be the projection $\mathbb{A}$ on $\mathbb{W}$. Then the closure $\overline{\mathbf{Pr}_{\mathbb{W}}(\mathbb{A})} = \mathbb{W}$.

Assume that $d = n$ and $\mathbb{W} = \mathbb{R}^n$. Then the system (2.9) is derived from the system (2.6) as a result of the transfer of origin from the point $\mathbf{0} \in \mathbb{R}^{2n}$ to the point $H(\mathbf{g}^*, \mathbf{h}^*) \in \mathbb{R}^{2n}$. If system (2.9) is chaotic (it means that system (2.6) is also chaotic), then its chaotic attractor must be located in the space of dimension more than $2n-1$. Consequently, after projection of the chaotic attractor from the space $\mathbb{R}^{2n}$ on the subspace $\mathbb{R}^n$, the attractor of system (2.9) got after projection can not be located in a subspace of dimension d, where $d \leq n - 1$. Thus, this attractor can not be regular and consequently it must be chaotic. (A chaotic attractor in $\mathbb{R}^n$ must have a fractional dimension between $n - 1$ and n.)

Further, since the magnitude $\|\mathbf{h}^*\|$ is small enough, then from existence of the chaotic attractor in system (2.10) it follows that the solutions of this system are bounded. It means that the solutions of system (2.9) (and system (2.6)) are also chaotic and bounded (the boundedness conditions are given in Theorem 5.3 [15]). Thus, if the attractor of system (2.6) is chaotic, then the attractor of system (2.4) must also be chaotic.

Introduce in system (2.4) the new variable $\mathbf{x}_r$ is given by the formula: $\mathbf{x} \to \mathbf{x}_r + \mathbf{g}^*$. Then system (2.4) can be transformed in the system

$$\dot{\mathbf{x}}_r(t) = (A + B(\mathbf{g}^*))\mathbf{x}_r(t) + \mathbf{r} + \begin{pmatrix} \mathbf{x}_r^T(t) B_1 \mathbf{x}_r(t) \\ \cdots \\ \mathbf{x}_r^T(t) B_n \mathbf{x}_r(t) \end{pmatrix}, \tag{2.11}$$

where the vector $\mathbf{r} \in \mathbb{R}^n$ depends on $\mathbf{h}^*$, and if $\mathbf{h}^* = 0$, then $\mathbf{r} = 0$ (in this case system (2.11) coincides with system (2.10)). Then the system (2.11) can be considered as a perturbed system with respect to system (2.10).

Further, from the condition $\|\mathbf{h}^*\| \to 0$ it follows that $\|\mathbf{r}\| \to 0$. It is well known that for any sufficiently small perturbation $\mathbf{r}$ the perturbed system (2.11) possesses that type of the chaos just as system (2.10). The proof is finished. □

Comment. Note that the existence of a chaotic attractor means that the solutions of both system (2.10) and system (2.4) are bounded. However, the existence of a chaotic attractor in system (2.4), generally speaking, does not imply the existence of a similar attractor in system (2.10); the solutions of system (2.10) can only be bounded.

Further, for simplification of denotations, we will use for the solutions of system (2.10) the symbol $\mathbf{x}(t)$ instead of $\mathbf{x}_d(t)$.

3. Autonomous Quadratic 3D Systems with signature (0,2)

It should be said that systems with signature $(l_{\mathbb{R}} > 0, l_{\mathbb{C}})$ are quite well studied. Only in very insignificant number of articles the nonlinear systems with signature $(0, l_{\mathbb{C}})$ were also investigated. Further we will consider the simplest class of such systems: it is the class of quadratic systems with signature $(0, 2)$.

At research of Lorenz-like and Chen-like attractors there is a situation when one of equations of describing the dynamics of the corresponding system is linear (see [4], [19]). One of such systems can be represented in the following way:

$$\begin{cases} \dot{x}(t) = p_{11}x(t) + p_{12}u(t) + p_{13}v(t) + a_1, \\ \dot{u}(t) = p_{21}x(t) + p_{22}u(t) + p_{23}v(t) + q_{11}x^2(t) + q_{12}x(t)u(t) + q_{13}x(t)v(t) + b_1, \\ \dot{v}(t) = p_{31}x(t) + p_{32}u(t) + p_{33}v(t) + r_{11}x^2(t) + r_{12}x(t)u(t) + r_{13}x(t)v(t) + c_1. \end{cases} \tag{3.1}$$

Here $p_{11}, ..., r_{13}, a_1, b_1, c_1 \in \mathbb{R}$ (the case $a_1^2 + b_1^2 + c_1^2 \neq 0$ is not excepted).

In [4] it is shown that by means of suitable linear changes of variables system (3.1) can be transformed to the following form

$$\begin{cases} \dot{x}(t) = \qquad\qquad h_{12}y(t) + h_{13}z(t), \\ \dot{y}(t) = h_{21}x(t) + h_{22}y(t) + h_{23}z(t) + x(t)z(t), \\ \dot{z}(t) = h_{31}x(t) + h_{32}y(t) + h_{33}z(t) - x(t)y(t) + px(t)z(t) + 1. \end{cases} \tag{3.2}$$

Here $h_{12}, ..., h_{33}, p \in \mathbb{R}$ and we suppose that $h_{12}^2 + h_{13}^2 \neq 0$. (In opposite case we get $h_{12} = h_{13} = 0$, $x(t) = const$, and system (3.2) is transformed in a linear system.)

With the help of Lemma 2.3 the answer on the question about existence of chaos in system (3.2) can be derived by research of its degenerate system. Such a study was carried out in article [4].

Let

$$\begin{cases} \dot{x}(t) = & f_{12}y(t) + f_{13}z(t), \\ \dot{y}(t) = f_{21}x(t) + f_{22}y(t) + f_{23}z(t) + x(t)z(t), \\ \dot{z}(t) = f_{31}x(t) + f_{32}y(t) + f_{33}z(t) - x(t)y(t) + px(t)z(t), \end{cases} \tag{3.3}$$

be the degenerative system for system (3.2).

Theorem 3.1. *[4] Suppose that for system (3.3) the following conditions:*

(a1) $f_{22} < 0$*;*

(a2) $|p| < 2$*;*

(a3) $f_{22}^2 + 4f_{12}f_{21} < 0$

are fulfilled. Then all solutions of system (3.3) are bounded.

The following theorem allows us to clarify some properties of the solutions of system (3.2).

Theorem 3.2. *Let* $\mathcal{A} \subset \mathbb{R}^3$ *be an attractor of system (3.2) (there are no equilibrium points in this system). Let* $\mathcal{B} \subset \mathbb{R}^3$ *denote the basin of attraction of this attractor:* $\mathcal{A} \subset \mathcal{B}$*.*

Suppose that for the degenerate system (3.3) in Theorem 3.1 the condition (a2) is replaced by condition $-2 < p < 0$ *(the remaining conditions are retained). Furthermore, we also assume that for system (3.2) the condition* $h_{12}(h_{31} - h_{21}p) - h_{13}h_{21} < 0$ *is satisfied.*

Then $\mathcal{B} = \mathbb{R}^3$ *and there exists a transport channel (this is a straight line* $\mathbb{L}$ *connecting points* $(0, h_{31} - h_{21}p, -h_{21})$ *and* $(c > 0, h_{31} - h_{21}p, -h_{21})$*) such that a trajectory starting from any point of region* $\mathbb{R}^3 - \mathcal{A}$ *will reach to the attractor* $\mathcal{A}$ *only along the straight line* $\mathbb{L}$*.*

Proof. According to Lemma 2.3, the boundedness of solutions of the degenerate system (3.3) implies the boundedness of solutions of system (3.2) .

Let us calculate the Jacobian matrix of system (3.2) at the point $(c, h_{31} - h_{21}p, -h_{21})$:

$$J(c) = \begin{pmatrix} 0 & h_{12} & h_{13} \\ 0 & h_{22} & h_{23} + c \\ 0 & h_{32} - c & h_{33} + pc \end{pmatrix}.$$

(a) $c > 0$. In this case, for a sufficiently large c, the matrix $J(c)$ has the following eigenvalues: $\lambda_1 = 0, \lambda_{2,3} = -r \pm qi$, where $r > 0, q > 0$.

Further, in a sufficiently small neighborhood of point $(c, h_{31} - h_{21}p, -h_{21})$ we have $\dot{x}(t) = h_{12}(h_{31} - h_{21}p) - h_{13}h_{21}$. Then for sufficiently small $\Delta t > 0$, we can write: $x(t) = c + (h_{12}(h_{31} - h_{21}p) - h_{13}h_{21})\Delta t$.

Thus, if $c > 0$ and $h_{12}(h_{31} - h_{21}p) - h_{13}h_{21} < 0$, then $x(t) \to \mathcal{A}$. As for $y(t), z(t)$, then due to the fact that $\Re(\lambda_{2,3}) = -r < 0$, we will have $y(t) \to h_{31} - h_{21}p$ and $z(t) \to -h_{21}$.

(b) $c < 0$. In this case $x(t)$ is a negative decreasing function. But since all solutions of system (3.2) are bounded, there is a moment t^* such that $x(t^*) > 0$. Now, taking point $(x(t^*), h_{31} - h_{21}p, -h_{21}))$ as the starting point, we reduce the proof to item (a).

Finally, from items (a) and (b) it follows that $|c|$ can be chosen arbitrarily large; this means that $\mathcal{B} = \mathbb{R}^3$. □

4. Systems with bounded solutions on the whole space $\mathbb{R}^n$

Consider the following autonomous system of differential equations

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + \cdots + a_{1n}x_n(t) + d_1 + \phi(x_2(t), ..., x_n(t)), \\ \dot{x}_2(t) = a_{21}x_1(t) + \cdots + a_{2n}x_n(t) + d_2 + x_1(t)(b_{22}x_2(t) + \cdots + b_{2n}x_n(t)), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots , \\ \dot{x}_n(t) = a_{n1}x_1(t) + \cdots + a_{nn}x_n(t) + d_n + x_1(t)(b_{n2}x_2(t) + \cdots + b_{nn}x_n(t)), \end{cases}, \tag{4.1}$$

where $a_{11} = 0$ and $\phi(x_2, ..., x_n)$ is a function of its arguments differentiable on the whole space $\mathbb{R}^{n-1}$ such that

$$\phi(0, ..., 0) = 0, \frac{\partial \phi(x_2, ..., x_n)}{\partial x_i}\Big|_{(0,...,0)} = 0; i = 2, ..., n.$$

Let $c \neq 0$ be an arbitrary real constant.

Along with system (4.1) , we will also consider the following system

$$\begin{cases} \dot{x}_2(t) = a_{21}c + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + d_2 + c \cdot (b_{22}x_2(t) + \cdots + b_{2n}x_n(t)), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots , \\ \dot{x}_n(t) = a_{n1}c + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + d_n + c \cdot (b_{n2}x_2(t) + \cdots + b_{nn}x_n(t)). \end{cases} \tag{4.2}$$

(This system is derived from system (4.1) after replacing $x_1(t) = c = const.$)

Theorem 4.1. *Assume that for any real c the point $E^*(c) = (x_2^*(c), ..., x_n^*(c))^T$ be a unique equilibrium point of system (4.2).*

Let's assume also that for system (4.2) the following conditions are satisfied:

(b1) $\lim_{c\to\infty} E^*(c) = \lim_{c\to-\infty} E^*(c) = E^*$, *where* $\|E^*\| \neq \infty$;

(b2) for all sufficiently large $c > 0$ the Jacobian matrix $J(c, E^*)$ of system (4.2) at point E^* is Hurwitz;

(b3) $\lim_{t\to\infty} \dot{x}_1(t) \to d_1 + (a_{12}, ..., a_{1n}) \cdot E^* + \phi(E^*) < 0$;

(b4) $\forall \mathbf{v} \neq \mathbf{0} \in \mathbb{R}^{n-1}$ *the function* $\phi(\mathbf{v}) > 0$ *on the open set* $\{\mathbb{R}^{n-1} - \mathbf{0}\}$.

Then all solutions of system (4.1) are bounded and they oscillate along the line $\mathcal{L}$ parallel to the x_1 axis and passing through the points $(c_b, E^*)^T \in \mathbb{R}^n$ and $(c_e, E^*)^T \in \mathbb{R}^n$, where c_b and c_e are any real numbers such that $c_b \neq c_e$.

Proof. Without loss of generality, we can assume that $n = 3$. (For an arbitrary $n > 3$, the proof of the theorem will be similar.) As a result, we get the following system

$$\begin{cases} \dot{x}(t) = \qquad a_{12}y(t) + a_{13}z(t) + d_x + \phi(y(t), z(t)), \\ \dot{y}(t) = a_{21}x(t) + a_{22}y(t) + a_{23}z(t) + d_y + x(t)(b_{22}y(t) + b_{23}z(t)), \\ \dot{z}(t) = a_{31}x(t) + a_{32}y(t) + a_{33}z(t) + d_z + x(t)(b_{32}y(t) + b_{33}z(t)). \end{cases} \tag{4.3}$$

If we now put $\dot{x}(t) = 0$ and $x(t) = c$, then system (4.3) is transformed into the following system

$$\begin{cases} \dot{y}(t) = a_{21}c + a_{22}y(t) + a_{23}z(t) + d_y + c \cdot (b_{22}y(t) + b_{23}z(t)), \\ \dot{z}(t) = a_{31}c + a_{32}y(t) + a_{33}z(t) + d_z + c \cdot (b_{32}y(t) + b_{33}z(t)). \end{cases} \tag{4.4}$$

Let $c \to \pm\infty$. In this case, the equilibrium point of system (4.4) will tend to the point

$$E^* = (y^*, z^*) = \Big(-\frac{b_{33}a_{21} - b_{23}a_{31}}{b_{33}b_{22} - b_{23}b_{32}}, \frac{b_{32}a_{21} - b_{22}a_{31}}{b_{33}b_{22} - b_{23}b_{32}}\Big),$$

where according to *(b1)* the norm $\|E^*\| \neq \infty$.

Let us represent the first equation of system (4.3) in the following integral form:

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} \Big(a_{12}y(\tau) + a_{13}z(\tau) + d_x + \phi(y(\tau), z(\tau))\Big) d\tau. \tag{4.5}$$

We will first consider the case $x(t_1) = c > 0$. Then, according to conditions *(b3)* and *(b2)*, from (4.5) it follows that if the time $t_2 > t_1$ is sufficiently large, then there exists a time $t^* \in (t_1, t_2)$ such that $x(t^*) = 0$ and at $t > t^*$, we have $x(t) < 0$. (Since system (4.4) is Hurwitz, then the vector $(y(t), z(t)) \to E^*$.)

Now let it be $x(t_1) = c < 0$. In this case system (4.4) ceases to be Hurwitz. Therefore, by virtue of condition *(b4)* of Theorem 4.1, the function $\phi(y(t), z(t)) > 0$ increases and if the time $t_2 > t_1$ is sufficiently large, then there exists a time $t^* \in (t_1, t_2)$ such that $x(t^*) = 0$ and at $t > t^*$, we have $x(t) > 0$.

Thus, we come to the following statement: if $x(t_1) > 0$ and $t \in (t_1, t^*)$, then $x(t) > 0$; if $x(t_1) < 0$ and $t \in (t_1, t^*)$, then $x(t) < 0$.

It is clear that at $t_1 \to t_2$ there is a transition through point t^* of the trajectory $(x(t), y(t), z(t))^T \in \mathbb{R}^3$ from region $x(t) > 0$ $(t_1 < t < t^*)$ to region $x(t) < 0$ $(t^* < t < t_2)$. Consequently, there exists a moment t_r such that at $t > t_r$ the trajectory $(x(t), y(t), z(t))^T$ will begin to move along the straight line $\mathcal{L}$ from point $(x(t_r) > 0, y(t_r), z(t_r))^T$ to point $(0, E^*)^T$. After passing through the point

$(0, E^*)^T$, there is again a moment $t_l > t_r$ such that at $t > t_l$ we get $(x(t_l) < 0, y(t_l), z(t_l))^T$.

Thus, we have oscillations of solution $(x(t), y(t), z(t))^T$ of system (4.3) in the neighborhood of point $(0, E^*)^T$ along the straight line $\mathcal{L}$. □

By $F_{\min}$ denote the minimum value of the function

$$F(x_2, ..., x_n) = a_{12}x_2 + \cdots + a_{1n}x_n + d_1 + \phi(x_2, ..., x_n)$$

(due to condition *(b4)*, the function $F(x_2, ..., x_n)$ is bounded from below).

Thus, in order to obviously guarantee the fulfillment of condition *(b3)*, it is necessary to consider the number d_1 to be negative and sufficiently large in absolute value.

Let us calculate Lyapunov's exponent $\Lambda[f]$ for a real function $f(t)$ [10]:

$$\Lambda[f] = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \left| \frac{f(t)}{f(t_0)} \right|, f(t_0) \neq 0. \tag{4.6}$$

Let $\Lambda[x_1], \Lambda[x_2], \ldots \Lambda[x_n]$ be all n Lyapunov's exponents of system (4.1).

Theorem 4.2. *Under the conditions of Theorem 4.1, among all Lyapunov's exponents of system (4.1), there exists at least one nonnegative exponent.*

Proof. From the first equation of system (4.1) it follows that

$$x_1(t) = x_1(0) + \int_0^t \Big(\sum_{i=2}^{n} a_{1i}x_i(\tau) + d_1 + \phi(x_2(\tau), ..., x_n(\tau)) \Big) d\tau.$$

Then, from here and (4.6), we have

$$\Lambda[x_1] = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \frac{1}{|x_1(0)|} \left| x_1(0) + \int_0^t \Big(\sum_{i=2}^{n} a_{1i}x_i(\tau) + d_1 + \phi(x_2(\tau), ..., x_n(\tau)) \Big) d\tau \right|.$$

Since the integrand is not less than $F_{\min}$, we have

$$\Lambda[x_1] \geq \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \frac{|x_1(0) + F_{\min}t|}{|x_1(0)|} \geq \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \Big| 1 + \frac{F_{\min}}{x_1(0)} t \Big| = 0.$$

The last inequality means that solutions of system (4.1) may exhibit chaotic behavior. □

5. Examples of quadratic systems with signatures (0,2), (2,2), and (0,4)

1. Consider the following system with signatures (0,2):

$$\begin{cases} \dot{x}(t) = 8.6y(t) - 15z(t), \\ \dot{y}(t) = -2.9x(t) - 3.9y(t) - 1.5z(t) + x(t)z(t)), \\ \dot{z}(t) = -0.2x(t) - 2.0y(t) + 1.45z(t) - x(t)y(t) - 0.24x(t)z(t) + 1, \end{cases} \tag{5.1}$$

In this case the complex equilibrium points can be found from the following algebraic system of equations:

$$8.6y-15z = 0, -2.9x-3.9y-1.5z+3xz = 0, -0.2x-2y+1.45z-xy-0.24xz+1=0.$$

From this we obtain two complex equilibrium points:

$$H_1 = \begin{pmatrix} x = -0.2749359445 + 1.148385374i \\ y = 0.2483530056 - 0.6439687636i \\ z = 0.1423890565 - 0.3692087578i \end{pmatrix},$$

$$H_2 = \begin{pmatrix} x = -0.2749359445 - 1.148385374i \\ y = 0.2483530056 + 0.6439687636i \\ z = 0.1423890565 + 0.3692087578i \end{pmatrix}.$$

Thus, we have $\mathbf{g} = (g_x, g_y, g_z)^T = (-0.27493594, 0.24835300, 0.14238905)^T$.

Construct the degenerate system (2.10). Here we have: $f_{12} = h_{12}, f_{13} = h_{13}, f_{21} = h_{21} + g_z, f_{22} = h_{22}, f_{23} = h_{23} + g_x, f_{31} = h_{31} - g_y + pg_z, f_{32} = h_{32} - g_x, f_{33} = h_{33} + pg_x$. Thus, we derive the following system:

$$\begin{cases} \dot{x}(t) = 8.6y(t) - 15z(t), \\ \dot{y}(t) = -2.7576x(t) - 3.9y(t) - 1.7749z(t) + x(t)z(t)), \\ \dot{z}(t) = -0.4825x(t) - 1.7251y(t) + 1.5080z(t) - x(t)y(t) - 0.24x(t)z(t). \end{cases} \tag{5.2}$$

(This system has one double real equilibrium point (0,0,0). The eigenvalues of the Jacobian matrix at point (0,0,0) are $(0.0000, -1.1958 \pm 2.4706i)$.)

It is easy to verify that for system (5.2) all the conditions of Theorem 3.1 are satisfied.

Now we return to system (5.1). Here we have $p = -0.24 < 0$, $h_{31} - h_{21}p = -0.896; -h_{21} = 2.900$ and $h_{12}(h_{31} - h_{21}p) - h_{13}h_{21} = -51.146 < 0$. All the conditions of Theorem 3.2 are satisfied. The results of the study of system (5.1) are presented in Fig. 6.1.

2. Consider the following system of signature $(2, 2)$:

$$\begin{cases} \dot{x}(t) = 8.6y(t) - 15.0z(t) + 0.24y(t)z(t) + 1.16z^2(t) + 0.75, \\ \dot{y}(t) = -2.9x(t) - 1.9y(t) - 1.5z(t) + 1.0x(t)z(t), \\ \dot{z}(t) = -0.2x(t) - 2.0y(t) + 1.0z(t) - 3.2x(t)y(t) - 0.24x(t)z(t) + 1.0. \end{cases} \tag{5.3}$$

From here we obtain four (two real and two complex) equilibrium points:

$$H_1 = \begin{pmatrix} x = -0.9447264757 \\ y = -36.76734204 \\ z = 29.69561519 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} x = 2.117189443 \\ y = 0.7533302783 \\ z = 12.26718473 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} x = -0.1563483580 - 0.5322550773i \\ y = 0.1903308266 + 0.5079267475i \\ z = 0.1519502360 + 0.3004214765i \end{pmatrix},$$

$$H_4 = \begin{pmatrix} x = -0.1563483580 + 0.5322550773i \\ y = 0.1903308266 - 0.5079267475i \\ z = 0.1519502360 - 0.3004214765i \end{pmatrix}.$$

The degenerate system is following:

$$\begin{cases} \dot{x}(t) = \quad 8.636y(t) - 14.602z(t) + 0.24y(t)z(t) + 1.16z^2(t), \\ \dot{y}(t) = -2.748x(t) - 1.9y(t) - 1.656z(t) + 1.0x(t)z(t), \\ \dot{z}(t) = -0.846x(t) - 1.499y(t) + 1.037z(t) - 3.2x(t)y(t) - 0.24x(t)z(t). \end{cases} \tag{5.4}$$

From here we obtain two zero equilibrium points H_1, H_2 at the origin and two real equilibrium points:

$$H_3 = \begin{pmatrix} x = -0.7818485908 \\ y = -36.64553597 \\ z = 29.43776844 \end{pmatrix}, H_4 = \begin{pmatrix} x = 2.236392419 \\ y = 0.4867712491 \\ z = 12.18973437 \end{pmatrix}.$$

In Fig. 6.2 the chaotic attractors of the system (5.3) and the system (5.4) (and its singular point) are shown.

3. Consider the following system of signature $(0, 4)$:

$$\begin{cases} \dot{x}(t) = -1.2x(t) - 1.0y(t) - 0.5z(t) - 0.4x(t)y(t) + 1.1y^2(t) \\ \qquad -1.0y(t)z(t) - 4.8, \\ \dot{y}(t) = 4.7x(t) + 1.2y(t) + 4.0z(t) - 1.1x(t)y(t) + 1.0x(t)z(t) \\ \qquad -0.4y^2(t) - 0.4z^2(t), \\ \dot{z}(t) = 2.0x(t) - 4.0y(t) + 1.0z(t) - 1.0. \end{cases} \tag{5.5}$$

From this we obtain four complex equilibrium points:

$$H_1 = \begin{pmatrix} x = -1.751955684 + 0.8534802106i \\ y = -0.7626493568 - 0.5093557404i \\ z = 1.453313941 - 3.744383383i \end{pmatrix},$$

$$H_2 = \begin{pmatrix} x = -1.751955684 - 0.8534802106i \\ y = -0.7626493568 + 0.5093557404i \\ z = 1.453313941 + 3.744383383i \end{pmatrix},$$
$$H_3 = \begin{pmatrix} x = 5.825941906 - 8.146158233i \\ y = 1.595614935 - 4.873503383i \\ z = -4.269424072 - 3.201697065i \end{pmatrix},$$
$$H_4 = \begin{pmatrix} x = 5.825941906 + 8.146158233i \\ y = 1.595614935 + 4.873503383i \\ z = -4.269424072 + 3.201697065i \end{pmatrix}.$$

The degenerate system is the following:

$$\begin{cases} \dot{x}(t) = -0.895x(t) - 3.430y(t) + 0.263z(t) - 0.4x(t)y(t) + 1.1y^2(t) \\ \qquad -1.0y(t)z(t), \\ \dot{y}(t) = 6.992x(t) + 3.737y(t) + 1.085z(t) - 1.1x(t)y(t) + 1.0x(t)z(t) \\ \qquad -0.4y^2(t) - 0.4z^2(t), \\ \dot{z}(t) = 2.0x(t) - 4.0y(t) + 1.0z(t). \end{cases} \tag{5.6}$$

(For the system (5.5) ((5.6)) the conditions of Theorem 3.2 (Theorem 3.1) are not satisfied.)

From here we obtain two zero equilibrium points H_1, H_2 at the origin and two complex equilibrium points:

$$H_3 = \begin{pmatrix} x = 7.577897589 - 6.958810395i \\ y = 2.358264291 - 4.163163179i \\ z = -5.722738013 - 2.735031924i \end{pmatrix},$$
$$H_4 = \begin{pmatrix} x = 7.577897589 + 6.958810395i \\ y = 2.358264291 + 4.163163179i \\ z = -5.722738013 + 2.735031924i \end{pmatrix}.$$

On Fig. 6.3 the chaotic attractors of the system (5.5) and the system (5.6) are shown.

6. Systems with antisymmetric matrices

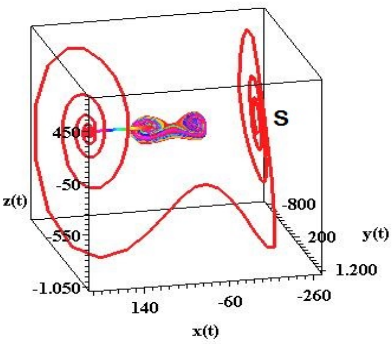
Now we consider the following application of Theorem 4.1.

Let us introduce the following restrictions in system (4.2): the matrices

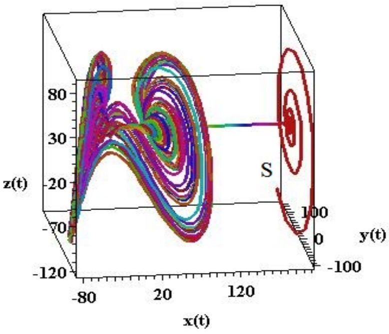
$$A_a = (a_{ij})_{i,j=2,...,n} - \text{diag}(a_{22}, ..., a_{nn}) \text{ and}$$
$$B_a = (b_{ij})_{i,j=2,...,n} - \text{diag}(b_{22}, ..., b_{nn})$$

are antisymmetric. It is obvious that if for any sufficiently large number $c > 0$ and $\forall i \in \{2, ..., n\}$, we have $a_{ii} + cb_{ii} < 0$, then the system (4.2) is stable. Let

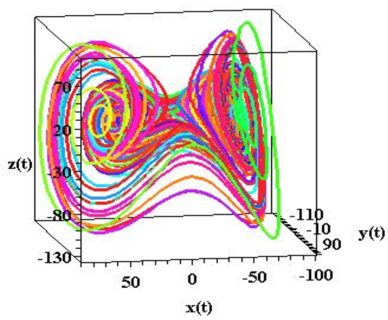
Phase portraits of systems discussed in sections 5 and 6



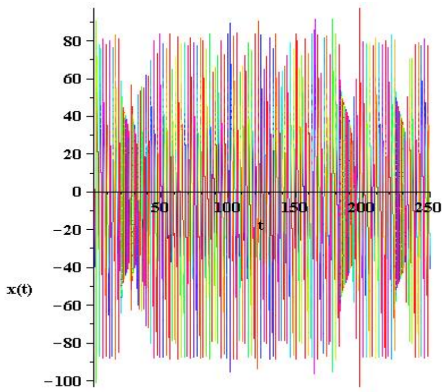
(a) S=($x_0 < 0, y_0, z_0$)



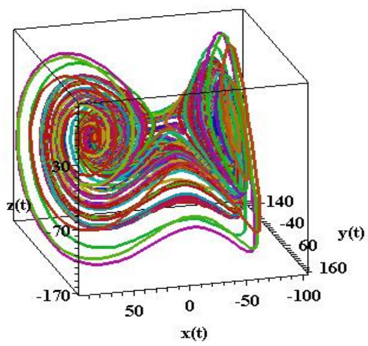
(b) S=($x_0 > 0, y_0, z_0$)



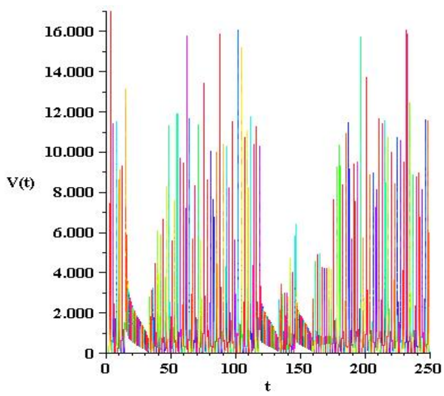
(c) S=($x_0 = 0, y_0, z_0$)



(d) S=($x_0 > 0, y_0, z_0$)

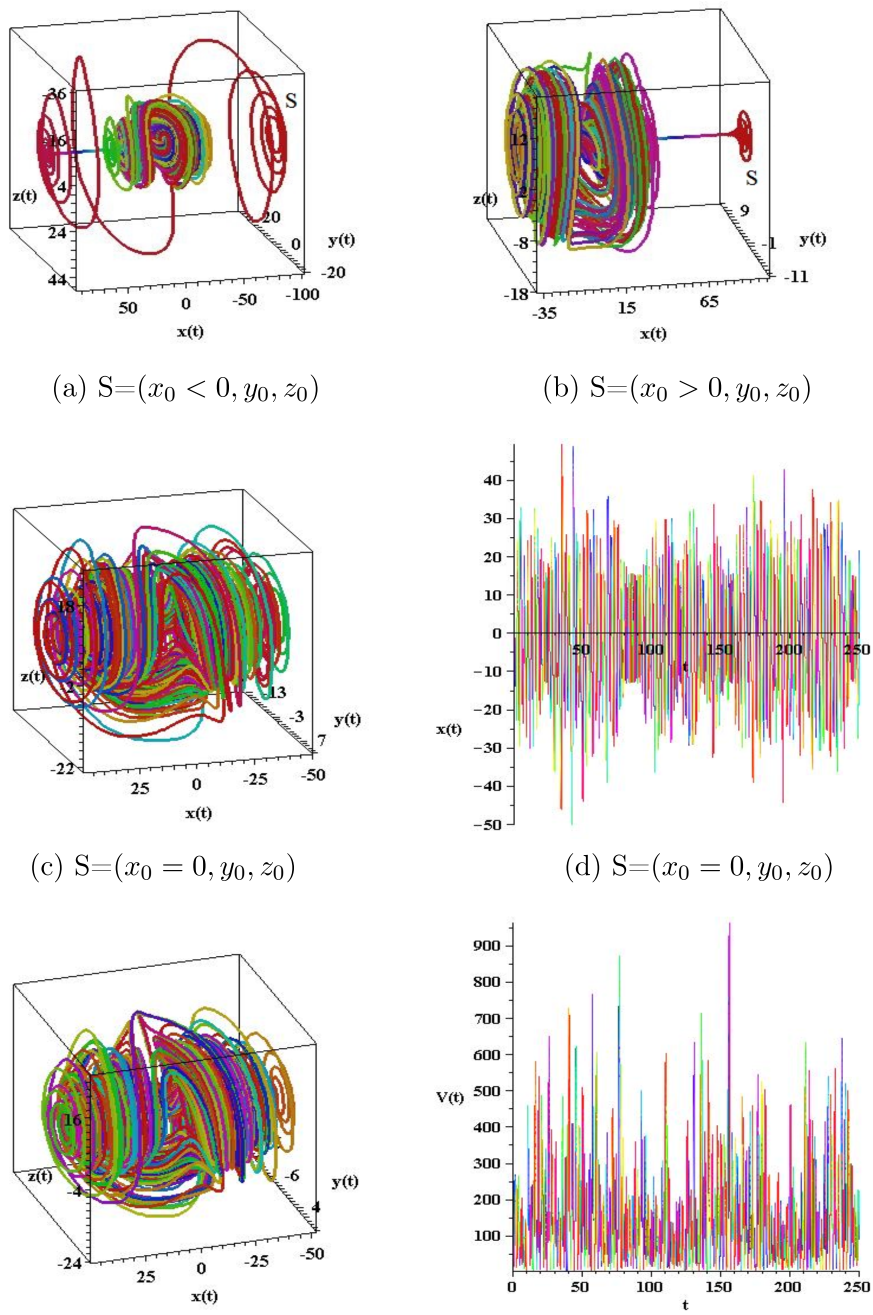


(e)



(f)

Fig. 6.1. Chaotic attractors of system (5.1) obtained at (a) $x_0 < 0$ in the starting point S, (b) $x_0 > 0$ in the starting point S and (c) $x_0 = 0$ in the starting point S; (d) behavior of solution $x(t)$ of system (5.1) in the starting point S; (e) chaotic attractor of the degenerate system (5.2) and (f) confirmation of its chaotic nature with the help of Lemma 2.3.



(a) S=$(x_0 < 0, y_0, z_0)$ (b) S=$(x_0 > 0, y_0, z_0)$

(c) S=$(x_0 = 0, y_0, z_0)$ (d) S=$(x_0 = 0, y_0, z_0)$

(e) (f)

Fig. 6.2. Chaotic attractors of system (5.3) obtained at (a) $x_0 < 0$ in the starting point S, (b) $x_0 > 0$ in the starting point S and (c) $x_0 = 0$ in the starting point S; (d) behavior of solution $x(t)$ of system (5.3) in the starting point S; (e) chaotic attractor of the degenerate system (5.4) and (f) confirmation of its chaotic nature with the help of Lemma 2.3.

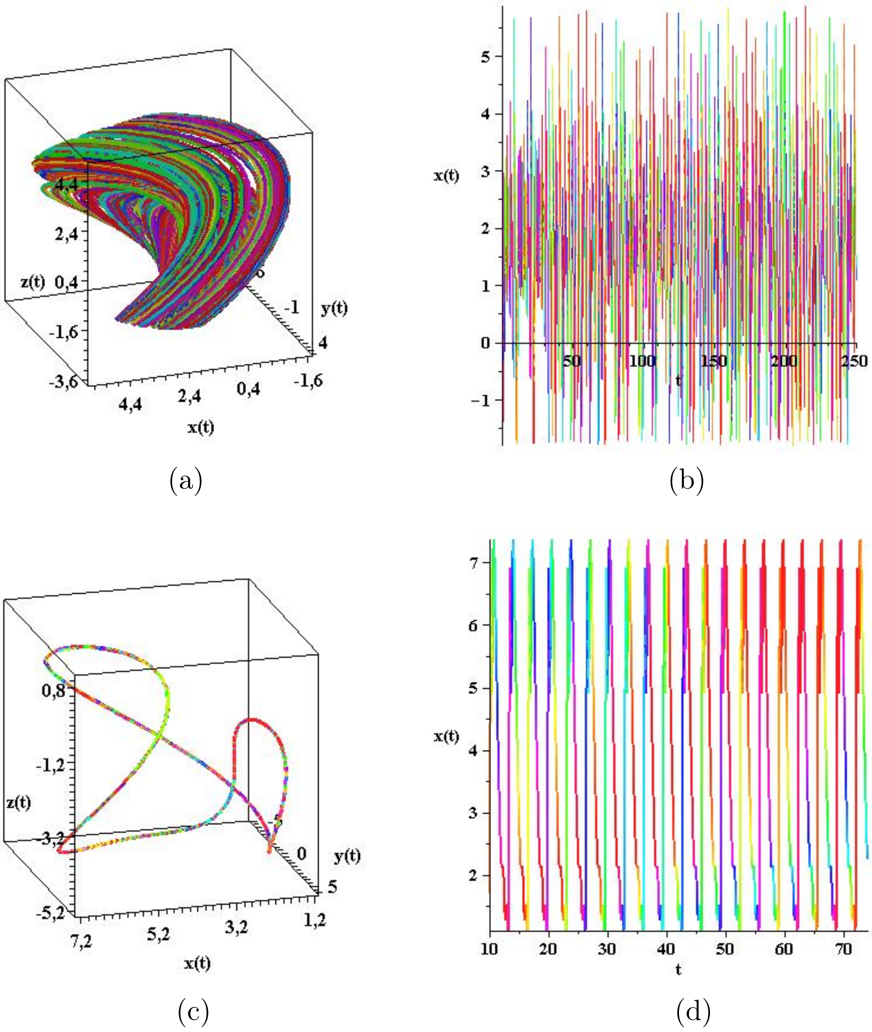


Fig. 6.3. Chaotic attractor (a) and the behavior of the solution $x(t)$ of system (5.5) (b). Limit cycle (c) and the behavior of the solution $x(t)$ of the degenerate system (5.6) (d)

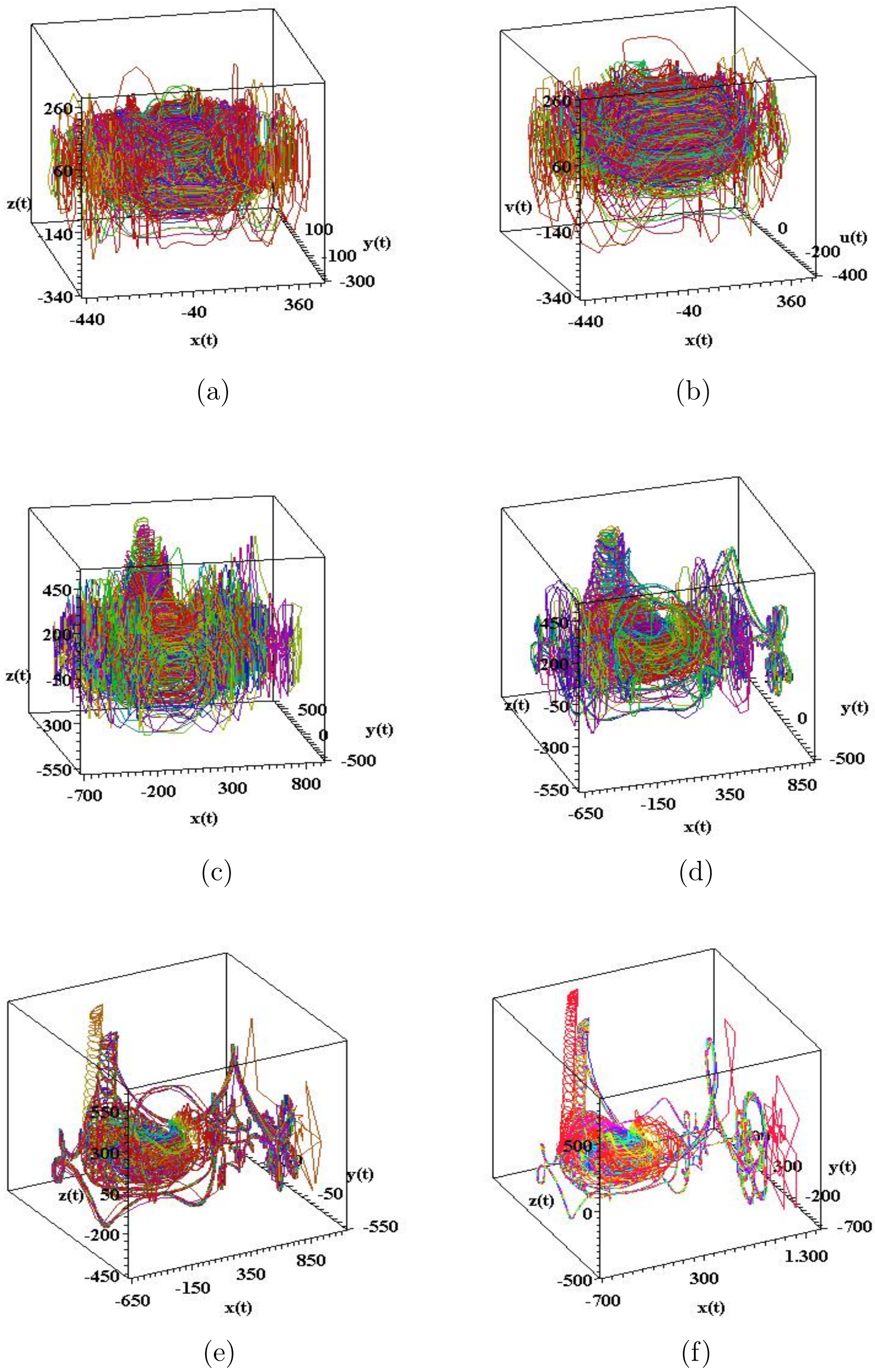


Fig. 6.4. Attractors of system (6.1) at: $a_{54} = -0.2, a = 0.15$(a); $a_{54} = -2, a = 0.15$ (b); $a_{54} = -2$, $a = -0.05$ (c); $a_{54} = -2, a = -0.125$ (d); $a_{54} = -2, a = -0.15$ (e); $a_{54} = -2, a = -0.2$ (f).

matrices A, B and vector $(d_1, ..., d_n)^T$ of the system (4.1) be as follows:

$$A = \begin{pmatrix} 0 & 1 & -13 & 0 & 0 \\ -41 & 0 & -3 & 0 & -1 \\ -2 & 3 & 0 & 1 & 1 \\ 340 & 0 & -1 & 0 & 2 \\ -11 & 1 & -1 & a_{54} & 0 \end{pmatrix},$$
$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1 & 1 & -1 & 0 \\ 0 & -1 & -0.1 & 0 & 2 \\ 0 & 1 & 0 & -0.1 & -1 \\ 0 & 0 & -2 & 1 & -0.1 \end{pmatrix}, \tag{6.1}$$

$d_1 = -11110, d_2 = 0, d_3 = -0.7, d_4 = 0, d_5 = 16$; a_{54} is a real parameter.

As for function $\phi(x_2, ..., x_n)$, we will write it as

$$\phi(x_2, ..., x_n) \equiv \phi(y, z, u, v) = ay^2 + 0.1yz + 0.16z^2 + 0.44u^2 + 0.2v^2,$$

where a is a real parametes.

On Fig. 6.4 the chaotic attractors of system (4.1) with matrices (6.1) are shown. (Note that in cases (c), (d), (e), and (f) the restrictions of Theorem 4.1 imposed on the function $\phi(y, z, u, v)$ are not satisfied: $a < 0$.)

7. Conclusion

In this paper, new types of attractors generated by systems of ordinary differential equations were obtained. The most sought-after applications of these attractors are in the fields of cybersecurity, communications, and cryptography. Therefore, we see our research continuing in the search for more complex attractors than those presented in this article.

In the future, we propose to strengthen Theorem 4.1 by introducing more complex functions $\phi(x_2, ..., x_n)$ into system (4.1). In addition, we also propose to replace the linear functions $b_{22}x_2+\cdots+b_{2n}x_n, ..., b_{n2}x_2+\cdots+b_{nn}x_n$ in system (4.1) with nonlinear functions $\phi_2(x_2, ..., x_n), ..., \phi_n(x_2, ..., x_n)$. This will significantly complicate the attractors generated by this system.

References

1. R. Bellman, *Introduction to Matrix Analysis – 2nd edition*, (Society for Industrial and Applied Mathematics 3600 University City Science Center/ Philadelphia, PA 19104–2688), 1997.
2. V. Ye. Belozyorov, *Exponential-algebraic maps and chaos in 3D autonomous quadratic systems*, International Journal of Bifurcation and Chaos, **25**, (2015), Article ID 1550048, 24 pages.
3. V. Ye. Belozyorov, S. A. Volkova, *Role of logistic and Ricker's maps in appearance of chaos in autonomous quadratic dynamical systems*, Nonlinear Dynamics, **83**(1) (2016), 719–729.

4. V. Ye. Belozyorov, *A novel search method of chaotic autonomous quadratic dynamical sytems without equilibrium points*, Nonlinear Dynamics, **86**(2),(2016), 835 – 860.
5. V. Ye. Belozyorov, *Invariant Approach to Existence Problem of Chaos in 3D Autonomous Quadratic Dynamical Systems*, International Journal of Bifurcation and Chaos, **26** (2016), Article ID 1650012, 14 pages.
6. V. Ye. Belozyorov, *On novel conditions of chaotic attractors existence in autonomous polynomial dynamical systems*, Nonlinear Dynamics, **91**(4) (2018), 2435–2452.
7. V. Ye. Belozyorov, *Universal approach to the problem of emergence of chaos in autonomous dynamical systems*, Nonlinear Dynamics, **95**(1)(2019), 579–595.
8. V. Ye. Belozyorov, D. V. Dantsev, *Modeling of chaotic processes by means of antisymmetric neural ODEs*, Journal of Optimization, Differential Equations and Their Applications (JODEA), **30**(1) (2022), 1–41.
9. V. Ye. Belozyorov, S. A. Volkova, *Odd and Even Functions in the Design Problem of New Chaotic Attractors*, International Journal of Bifurcation and Chaos, **32** (2022), Article ID 2250218, 26 pages.
10. V. Ye. Belozyorov, S. A. Volkova, *Discrete processes and chaos in systems of ordinary differential equations*, Journal of Optimization, Differential Equations and Their Applications (JODEA), **30**(2) (2022), 62–78.
11. V. Ye. Belozyorov, Y. V. Koshel, *On systems of neural ODEs with generalized power activation functions*, Journal of Optimization, Differential Equations and Their Applications (JODEA), **32**(2)(2024), 56–91.
12. V. Ye. Belozyorov, N. A. Guk, D. I. Yehoshkin, *Encryption of color images based on chaotic attractors generated by ODE systems containing module nonlinearities*, Journal of Optimization, Differential Equations and Their Applications (JODEA), **32**(2)(2024), 92–117.
13. V. Ye. Belozyorov, D. M. Moroz, S. A. Volkova, *Homoclinic and heteroclinic neural ODEs: theory and its use to construct new chaotic attractors*, Journal of Optimization, Differential Equations and Their Applications (JODEA), **33**(1) (2025), 15–41.
14. L. Boi, *Geometry of dynamical systems and topological stability: from bifurcation, chaos and fractals to dynamics in natural and life scienses*, International Journal of Bifurcation and Chaos, **21** (2011), 815 – 867.
15. H. K. Khalil, *Nonlinear Systems – 2nd Edition*, (Prentice Hall/ Upper Saddle River, New-Jersy 07458), 1996.
16. D. Mumford, *Algebraic Geometry. Complex projective varieties*, (Springer-Verlag/ Berlin, Heidelberg, New York), 1976.
17. G. Teschi, *Ordinary Differential Equations and Dynamical Systems*, Graduate Studies in Mathematics, **140** (AMS, Providence/ Rhode Island), 2012.
18. Z. Wang, S. Cang, E.O. Ochola, Y. Sun, *A hyperchaotic system without equilibrium*, Nonlinear Dynamics, **69** (2012), 531 – 537.
19. X. Wang, G. Chen, *A gallery Lorenz-like and Chen-like attractors*, International Journal of Bifurcation and Chaos, **23** (2013), Article ID 1330011, 20 pages.

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