Cracks Interaction in the Elastic Composite under Action of the Harmonic Loading Field

V. Z. Stankevych
Department of Computational Mechanics of Deformable Systems
Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, NASU
Lviv, Ukraine
stan_volodja@yahoo.com

I. O. Butrak
Department of Computational Mechanics of Deformable Systems
Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, NASU
Lviv, Ukraine
butrak@ukr.net

V. V. Kovalchuk
Department of Rolling Stock and Track
Lviv Branch of Academician V. Lazaryan Dnipropetrovsk National University of Railway Transport
Lviv, Ukraine
kovalchuk.diit@gmail.com

***Abstract*—A three-layer composite with penny-shaped cracks in the field of harmonic torsional loading is considered. The solutions are chosen in the form of Helmholtz potentials with densities that characterize unknown crack opening functions. The problem is reduced to the solution of the system of two-dimensional boundary integral equations (BIEs). The influence of the frequency of the applied load, the ratio of the elastic constant parameters of the composite on the dynamic stress intensity factors in the defect vicinity is investigated.**

***Index Terms*—penny-shaped cracks; three-layer composite; harmonic torsional loading; boundary integral equation method**

I. Introduction

Layered composites have wide applications in a variety of industries – from aerospace technology to medicine. By a combination of materials with different characteristics the new materials with high mechanical properties, ability to work in aggressive environments, or which can take the unique optical properties, ets. can be obtained. Important interest represents the study of these materials on sturdiness under the influence of static and dynamic loadings [1-4]. The presence in the composites of the structural defects like cracks, cavities, alien inclusions requires a separate study because the defects above play a role of the stress concentrators and can initiate the solids destruction [5-11]. Dynamic character of loading leads to a complex interference of wave picture in the solid and generating the new types of elastic waves and dispersive phenomena. It affects on the time behavior of stress intensity factors (SIF) near crack contour. SIF can significantly greater than their same static analogues that is danger in terms of loss of design integrity.

The BIEs method is successfully using for investigation of dynamic problems of layered composites with cracks [12-15]. The BIEs method for investigation of stress-deformable state of three-layer solid with circular cracks in time-harmonic torsional loading field is applied in this paper.

II. Prroblem Statement

Let's consider the elastic composite, which consists a layer L with thickness h. Composite is limited by the two parties half-spaces A. Solid materials are isotropic and characterized by shear moduli G_D, Poisson's ratios μ_D, and the mass densities ρ_D, $D = L, A$. Ideal mechanical contact conditions are performed on conjugated interface surfaces S_m, $m = 1, 2$. Half-spaces contain a penny-shaped cracks with radius a, occupy surfaces S, parallel to interface surfaces and are located on the depths d. Opposite the cracks-surfaces $S^{\pm}$ are subjected to the action of self-balanced harmonic torsional loading as functions of time t with frequency ω

$$N_l^{+}(\mathbf{x},t) = -N_l^{-}(\mathbf{x},t) = (-1)^{l+1} x_{3-l}(1-\delta_{l3}) N_0 \exp(-i\omega t),$$
$$\mathbf{x}(x_1, x_2, x_{3j}) \in S_j, \quad l = \overline{1,3}, \quad j = 1,2.$$

Here $i = \sqrt{-1}$; $N_0 = const$ is the amplitude value of the applied loading; δ is the Kronecker delta.

In the domains S let choose the Cartesian coordinate systems $O_m x_1 x_2 x_{3m}$, $m = 1, 2$. In the median surface of layer let choose a coordinate system with origin, which is located on the same line (Fig. 1). All parameters of wave field in composite is characterized by harmonic time dependence. In composite the only elastic SH-wave with the specified location method of cracks and their loading is propagated. The problem for determining of stress-strain state of solid with cracks is reduced to problem of determining the amplitude values of component of elastic displacement vectors $\mathbf{u}^{(D)}(u_1^{(D)}, u_2^{(D)}, u_3^{(D)})$, $D = L, A$, that satisfy differential balance equations

$$\Delta_3 \mathbf{u}^{(D)} + k_2^{2(D)} \mathbf{u}^{(D)} = 0 \ , \ D = L, A \ . \tag{1}$$

Here $k_i^{(D)} = \omega / c_i^{(D)}$ is wave number; $c_2^{(D)} = \sqrt{G_D/\rho_D}$ is the

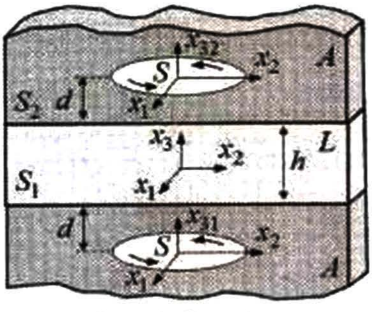

Figure 1. Problem's scheme

transverse elastic wave velocity for the D-the component of a solid; Δ_3 is the three-dimensional Laplace operator. To satisfy the equation (1), we formulate two groups of boundary conditions of the problem. In the domains S of cracks location

$$\sigma_{j3}(\mathbf{x}) = (-1)^{j+1} x_{3-j} N_0 \ , \quad \mathbf{x}(x_1, x_2, x_{3k}) \in S \ , \ j,k = 1,2 \ . \tag{2}$$

The second group of mixed boundary conditions formulated on the interface surfaces S_j, $j = 1,2$ of the conjugated component of the solid for tangential stresses, shear displacements and characterizes a perfect mechanical contact

$$\begin{aligned} & u_j^{(L)}(x_1, x_2, -h/2) = u_{j1}^{(A)}(x_1, x_2, d) \ , \\ & \sigma_{j3}^{(L)}(x_1, x_2, -h/2) = \sigma_{j31}^{(A)}(x_1, x_2, d) \ , \\ & u_j^{(L)}(x_1, x_2, h/2) = u_{j2}^{(A)}(x_1, x_2, -d) \ , \\ & \sigma_{j3}^{(L)}(x_1, x_2, h/2) = \sigma_{j32}^{(A)}(x_1, x_2, -d) \ , \ j = 1,2 \ . \end{aligned} \tag{3}$$

In this case, vertical displacements and normal stresses don't exist.

III. Method of Solution

The formulated dynamic problem of elasticity theory is determined by the BIEs method. The total wave field in an elastic layer is given as a sum

$$\mathbf{u}^{(L)} = \mathbf{u}_1^{(L)} + \mathbf{u}_2^{(L)} \ , \tag{4}$$

in elastic half-surfaces

$$\mathbf{u}_j^{(A)} = u_j^{(A)} + \mathbf{u}^{(A)} \ , \ j = 1,2 \ . \tag{5}$$

Displacements $\mathbf{u}_j^{(D)}$, $j = 1,2$, $D = L, A$ generated by the oscillation of interface surfaces $S_j^{(D)}$ points; displacements $\mathbf{u}^{(A)}$ are caused by the opening of opposite crack-surfaces $S^{\pm}$ under loading.

Displacements $u_{jl}^{(A)}$ from the crack opening can be represented in the form of Helmholtz potentials

$$\mathbf{u}_{jl}^{(A)}(\mathbf{x}) = \frac{\partial P_{jl}^{(A)}(\mathbf{x}_l)}{\partial x_{3l}} \ , \ j,l = 1,2 \ , \tag{6}$$

$$P_{jl}^{(A)}(\mathbf{x}_{lr}) = \iint_S \Delta u_{jl}(\xi) \frac{\exp\left[i k_2^{(A)} |\mathbf{x}_l - \xi| \right]}{|\mathbf{x}_l - \xi|} dS_\xi \ , \ \xi(\xi_1, \xi_2, 0) \in S$$

with the unknown densities characterizing the point displacements of opposite crack-surfaces in the direction $O_k x_{kl}$

$$\Delta u_{jl}(\mathbf{x}_l) = \left[u_{jl}(x_1, x_2, -0) - u_{jl}(x_1, x_2, +0) \right] / 4\pi \ , \ \mathbf{x}_l \in S_l \ .$$

In the considered method of cracks location in the solid and their loading, there are no normal crack openings, namely $\Delta u_{3l}(\mathbf{x}_l) = 0$, $l = 1,2$. Similarly, the displacements are chosen in the form

$$\mathbf{u}_{mj}^{(D)}(\mathbf{x}) = \frac{\partial P_{mj}^{(D)}(\mathbf{x}_l)}{\partial x_3} \ , \ m, jl = 1,2 \ , \ D = L, A \ , \tag{7}$$

$$P_{lj}^{(D)}(\mathbf{x}_{lr}) = \iint_{S_l} \alpha_{lj}^{(D)}(\xi) \frac{\exp\left[i k_2^{(D)} |\mathbf{x}_l - \xi| \right]}{|\mathbf{x}_l - \xi|} dS_\xi \ , \ \xi(\xi_1, \xi_2, 0) \in S_l$$

with unknown densities $\alpha_{lj}^{(D)}$ characterizing the displacement of points of interface surfaces S_l in the direction Ox_j.

By applying to (4), (5) the ratio of the Hooke's law, by considering (6), (7), a representation for the amplitude values of the stress tensor components are obtained

$$\begin{aligned} & \sigma_{j3}^{(L)} = \sigma_{j31}^{(L)} + \sigma_{j32}^{(L)} \ , \\ & \sigma_{j3l}^{(A)} = \sigma_{j3l}^{(A)} + \sigma_{j3l}^{(A)} \ , \ j,l = 1,2 \ , \\ & \sigma_{j3l}^{(D)}(\mathbf{x}) = -G_L(\Delta_2 + k_2^{(D)2}) P_{jl}^{(D)}(\mathbf{x}) \ , \ D = L, A \ , \\ & \sigma_{j3}^{(A)}(\mathbf{x}) = -G_A(\Delta_2 + k_2^{(A)2}) P_{jl}^{(A)}(\mathbf{x}) \ , \end{aligned} \tag{8}$$

where Δ_2 is the two-dimensional Laplace's operator.

Having satisfied the boundary conditions (3) of the problem, and considering representations (4) – (8), a system of eight two-dimensional BIEs for unknown densities Δu_{jl} and $\alpha_{jl}^{(D)}$ is obtained. Applying to the last two-dimensional Fourier integral transform on variables x_1, x_2, a system of linear algebraic equations (LAE) with respect to the Fourier transformant $\Delta\tilde{u}_{jl}$, $\tilde{\alpha}_{jl}^{(D)}$ is obtained. Having solved LAE, a representation $\tilde{\alpha}_{jl}^{(D)}$ via $\Delta\tilde{u}_{jl}$ is obtained. After applying the inverse two-dimensional integral Fourier transform, the representation has the form

$$\alpha_{jl}^{(A)}(\eta)=\frac{1}{4\pi^2}\int\int_{-\infty}^{+\infty}\int_0^{\infty}\frac{\tau\Delta u_{jl}(\mathrm{x}(\eta))\Omega(\tau)}{R(\tau)}J_0(\tau|\varsigma-\eta|)\,d\tau\, dS_\varsigma\ ,$$

$$\Omega(\tau)=(G_A^2 R_2^{(A)2}-G_L^2 R_2^{(L)2})(1-H_L^2)-4G_A G_L R_2^{(A)} R_2^{(L)} H_L\ ,$$

$$H=e^{-hk_2^{(L)}}\ ,\ R_2^{(D)}(\tau)=\sqrt{\tau^2-k_2^{(D)2}}\ ,\ D=L,A\ , \qquad (9)$$

$$R(\tau)=\left(G_L R_2^{(L)}+G_A R_2^{(A)}\right)^2-\left(G_L R_2^{(L)}-G_A R_2^{A}\right)^2 H_L^2\ .$$

Here $J_0(z)$ is the Boussel function of the zero order of a real argument z; the function $R(\tau)$ arises as a satisfaction result of the boundary conditions on the interface composite surfaces and characterizes the possibility of appearance in the solid of surface waves and associated dispersion phenomena.

The boundary conditions (2) satisfy at the last stage of solving the problem. Considering the representations (8), (9) and calculating integrals over infinite integration domains S_l, $l=1,2$

$$\int\int_{-\infty}^{+\infty}\frac{\exp\left[ik_2^{(D)}|\mathrm{x}-\varsigma|\right]}{|\mathrm{x}-\varsigma|}J_0(\tau|\varsigma-\eta|)\,dS_\varsigma=$$

$$=2\pi\frac{\exp\left[-|x_3|R_2^{(D)}(\tau)\right]}{R_2^{(D)}(\tau)}J_0(\tau|\bar{\mathrm{x}}-\eta|)$$

$$|\mathrm{x}-\varsigma|=\sqrt{(x_1-\varsigma_1)^2+(x_1-\varsigma_1)^2+x_3^2}\ ,$$

$$|\bar{\mathrm{x}}-\eta|=\sqrt{(x_1-\eta_1)^2+(x_1-\eta_1)^2}\ ,\ |\varsigma-\eta|=\sqrt{(\varsigma_1-\eta_1)^2+(\varsigma_1-\eta_1)^2}$$

the original problem is reduced to the solving of system two two-dimensional BIEs of Helmholtz potential type

$$\left(\Delta_2+k_2^{(A)2}\right)\iint_S \Delta u_j(\xi)\frac{\exp\left[ik_2^{(A)}|\mathrm{x}-\xi|\right]}{|\mathrm{x}-\xi|}dS_\xi+ \qquad (10)$$

$$+\iint_S \Delta u_j(\xi)\int_0^{\infty}\tau R_2^{(A)}(\tau)\frac{\Omega(\tau)}{R(\tau)}\exp\left|-2dR_2^{(A)}\right|J_0(\tau\rho)\,d\tau\, dS_\xi=$$

$$=(-1)^{j+1}x_{3-j}N_0\ ,\ j=1,2\ ,\ \mathrm{x}\in S\ ,\ \rho=\sqrt{(x_1-\xi_1)^2+(x_1-\xi_1)^2}\ .$$

The crack-opening functions Δu_j are unknown values of BIEs. Integration into (10) is defined only over the finite crack domains S, which is important for the numerical solving of the equation. Having developed of the first term of the BIEs in a series in power $|\mathrm{x}-\xi|$ for $k_2^{(A)}\to 0$, can show that the obtained BIEs contain a singularity of type $|\mathrm{x}-\xi|^{-3}$. Further regularization and numerical solving of BIEs are described in [9]. At the same time, unknown crack-opening functions can be written as

$$\Delta u_j\left(\mathrm{x},k_2^{(A)}\right)=\sqrt{a^2-x_1^2-x_2^2}\,\beta_j\left(\mathrm{x},k_2^{(A)}\right)\ ,\ j=1,2\ , \quad (11)$$

where β_j is unknown, twice continuous-differential in S function. The representations (11) mean that there are no jumps of displacement across the crack contour. During numerical solving of BIEs (10), the circular integration domain S was covered by a grid of quadrangular boundary elements in the polar coordinate system $Or\varphi$, the discrete values β_j were fixed within each of them. The division density of domain S was 20 elements per radial and 24 elements at angular coordinates. BIEs of the problem were reduced to the solving of the LAE system for discrete values β_j. The values of the latter in the contour elements of the domain S were determined the amplitude values of the mode-III dynamic stress intensity factor (SIF) by the relations

$$K_{III}^{d}\left(\varphi,k_2^{(A)}\right)=-2G_A\pi\sqrt{\pi a}\cdot$$

$$\cdot\left[\beta_1\left(a,\varphi,k_2^{(A)}\right)\sin\varphi-\beta_2\left(a,\varphi,k_2^{(A)}\right)\cos\varphi\right]\ .$$

IV. Results and Discussion

The dependences of the normalized amplitude values of the dynamic SIF $\bar{K}_{III}^{d}=\left|K_{III}^{d}\right|/K_{III}^{st}$ ($K_{III}^{st}=\frac{4}{3}N_0\sqrt{a/\pi}$ is the mode-III static SIF for a twisting crack in an infinite solid) versus the dimensionless wave number $k_2^{(A)}a$ for the case $G=G_A/G_L\le 1$ are presented in Fig. 2. As the frequency $k_2^{(2)}a$ increases dynamic SIF increase monotonically from their static values for $k_2^{(2)}a=0$, reach the maximum value, and then fall. The increase of the layer stiffness leads to an increase of absolute maxima $\bar{K}_{III}^{d}$ and an increase of the frequency $k_2^{(2)}a$ at which these maxima are reached. The cases $G=1$ (square-marked curve) and $G=0.01$, $G=0$ (circle-marked curve) characterize respectively the interaction of two cracks in an infinite solid and the case of a crack in a half-space with a clamped surface.

In Fig. 3*a*, *b* the dependences of the absolute maxima of the dynamic SIF amplitudes are presented in the frequency spectrum for the layer thickness h (at a fixed depth $d=0.5a$ of cracks occurrence in the half-spaces) and the depth d (at a fixed layer thickness $h=a$). The increase in the thickness of the layers leads to the gradual decrease of the maxima $\bar{K}_{III}^{d\max}$ and their propagation to their analogs for the case of a crack in a bimaterial, consisting of two half-spaces. At the same time, the decrease of the parameter G leads to a decrease the

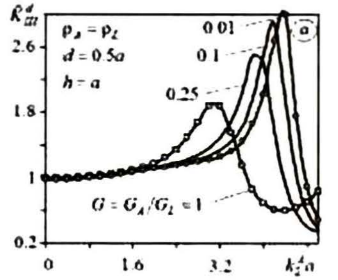

Figure 2. Dependences of the amplitudes of the dynamic SIF $\tilde{K}_{III}^{d}$ on the wave number $k_2^A a$

oscillating dependence of the dynamic SIF from h. The increase the depth of the cracks occurrence in the half-space leads to a decrease $\tilde{K}_{III}^{d\,\max}$ and their propagation to their analogues for the case of a single crack in an infinite solid [16]. For $d \geq 7a$ the considered curves do not depend on the parameter G.

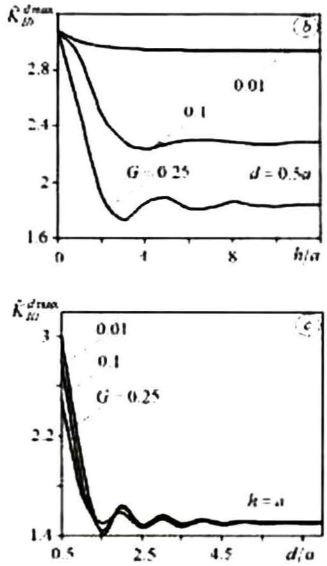

Figure 3. Dependences of the amplitudes of the dynamic SIF $\tilde{K}_{III}^{d}$ and $\tilde{K}_{III}^{d\,\max}$

ACKNOWLEDGMENT

This work was supported by the Scientific and Technology Center in Ukraine and the National Academy of Sciences of Ukraine (Project No. 6247), which are gratefully acknowledged.

REFERENCES

[1] H. Altenbach, J. Altenbach, and W. Kissing, "Mechanics of composite structural elements," Springer Verlag, 2018

[2] S. Gopalakrishnan, "Wave propagation in materials and structures," CRC Press, 2016

[3] B. Kennett, "Seismic wave propagation in stratified media," ANU E Press, 2009

[4] I. Turchyn, and O. Turchyn, "Transient plane waves in multilayered half-space," Acta Mechanica et Automatica, vol. 7, pp. 53-57, 2013.

[5] V. V. Matus, and Ya. I. Kunets, "Null field method of SH-wave scattering by partially debonded elastic inclusion," Proc. XIII Int. Seminar Workshop "Direct and inverse problems of electromagnetic and acoustic wave theory" (DIPED-2008), Lviv-Tbilisi, pp. 176-178, 2008

[6] M. Rekik, M. Neifar, and S. El-Borgi, "An axisymmetric problem of an embedded crack in a graded layer bonded to a homogeneous half-space," Int. J. of Solids and Struct., vol. 47, pp. 2043-2055, 2010

[7] Y. Protserov, and N. Vaysfeld, "Torsion problem for elastic multilayered finite cylinder with circular crack," Appl. Math. and Mech., vol. 38, pp. 423–438, 2017.

[8] M. Ryvkin, and J. Aboudi, "Stress redistribution due to cracking in periodically layered composites," Eng. Fract. Mech., vol. 93, pp. 225-238, 2012

[9] V. Z. Stankevych, "Stress intensity near a crack in the composition of a half space and a layer under harmonic loading," Materials Science, vol. 44, pp. 175-182, 2008.

[10] V. V. Mikhas'kiv, I. O. Butrak, and I. P. Laushnik, "Interaction between a compliant disk-shaped inclusion and a crack upon incidence of an elastic wave," J. of Appl. Mech. and Tech. Phys., vol. 54, pp. 465-471, 2013

[11] B. L. Wang, J. C. Han, and S. Y. Du, "Fracture mechanics for multilayers with penny-shaped cracks subjected to dynamic torsional loading," Int. J. of Eng. Science, vol. 38, pp. 893-901, 2000

[12] G. D. Manolis, and P. S. Dineva, "Elastic waves in continuous and discontinuous geological media by boundary integral equation methods. A review," Soil Dynamics and Earthquake Engineering, vol. 70, pp. 11-29, 2015

[13] M. V. Menshykova, O. V. Menshykov, I. A. Guz, M. Wuensche, and Ch. Zhang, "A boundary integral equation method in the frequence domain for cracks under transient loading," Acta Mechanika, vol. 227, pp. 3305-3314, 2016.

[14] V. Skalsky, O. Stankevych, and O. Serhiyenko, "Wave displacement field at a half-space surface caused by a internal crack under twisting load," Wave Motion, vol. 50, pp. 326-333, 2013

[15] V. Z. Stankevych, and M. V. Khai, "Study into the interaction of cracks in an elastic half-space under a shock load by means of boundary integral equations," Int. Appl. Mech, vol. 38, pp. 440-446, 2002

[16] E. P. Chen, and G. C. Sih, "Scattering waves about stationary and moving cracks," in. Mechanics of Fracture (G. C. Sih, ed.), vol. 4, Elastodynamic Cracks Problems – Leyden Noordhoff Int. Publ, pp. 119-212, 1977