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Refinement of the Maxwell formula for composite reinforced by circular cross-section fibers. Part I: using the Schwarz alternating method

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Abstract The effective properties of the fiber-reinforced composite materials with fibers of circular cross section are investigated. The novel estimation for the effective coefficient of thermal conductivity refining the classical Maxwell formula is derived. The method of asymptotic homogenization is used. For analytical solution of the periodically repeated cell problem, the Schwarz alternating process is employed. The principal term of the refined formula coincides with the classical Maxwell formula. On the other hand, the refined formula can be used far beyond the area of applicability of the Maxwell formula. It can be used for dilute and non-dilute composites. It is confirmed by comparison with known numerical and asymptotic results.

1 Introduction

In the case of a composite structure with periodically located cylindrical inclusions of circular cross sections, a formula for the heat transfer parameter obtained based on the three phase model (TPhM) of a composite coincides with the well-known Maxwell formula (MF) [1], and it reads

$$q = \frac{\lambda\left(1+\frac{\pi a^2}{4}\right)+1-\frac{\pi a^2}{4}}{\lambda\left(1-\frac{\pi a^2}{4}\right)+1+\frac{\pi a^2}{4}}. \tag{1.1}$$

Equation (1.1) is also referred to as the Maxwell–Garnett, Maxwell–Odelevskii, Clausius–Mossotti, Lorenz–Lorentz, Landauer, and Wiener–Wagner formula (see references [2–6]). A more detailed analysis of the MF implies the following conclusions:

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(i) relation (1.1) describes adequately a character of the effective composite conductivity in the case of the inclusions concentration $a \ll 1$ for arbitrary values of their conductivity λ;
(ii) for large sizes of inclusions $a \to 1$, the TPhM yields reliable results in the case of $\lambda \sim 1$;
(iii) TPhM does not work properly (qualitatively and quantitatively) for large sizes of inclusions $a \to 1$ for the limiting large ($\lambda \to \infty$) or limiting small ($\lambda \to 0$) conductivity. It means from the physical point of view that formula (1.1) offers neither a qualitative nor quantitative reliable description of the processes occurring in a composite, like the emergence of an infinite cluster. From the mathematical point of view, it refers to a lack of convergence of the asymptotic relation yielded by (1.1) for $\lambda \to \infty$, $a \to 1$, with regard to the asymptotic results [7]:

$$q_{\text{as}}^{(\infty)} = \frac{\pi}{\sqrt{1-a^2}} - \pi + 1 \approx \frac{3.14159}{\sqrt{1-a^2}} - 2.14159 \ for \lambda \to \infty, a \to 1. \tag{1.2}$$

The analogous conclusion holds for the case of limiting small conductivity of inclusions ($\lambda \to 0$) having large geometric sizes ($a \to 1$), where the Keller theorem implies the following estimation [8]:

$$q_{\text{as}}^{(0)} = \frac{\sqrt{1-a^2}}{\pi - (\pi - 1)\sqrt{1-a^2}} \text{ for } \lambda \to 0, a \to 1. \tag{1.3}$$

The asymptotic formula (1.2) was improved in [9] using the method of functional equations:

$$q_{\text{as}}^{(\infty)}(f) = \frac{5.56250}{\sqrt{\pi}\sqrt{1-a^2}} - 1.84934 \approx \frac{3.13830}{\sqrt{1-a^2}} - 1.84934,$$
$$\text{as } \lambda \to \infty,\ f \to f_c,\ \text{where } f = \frac{\pi a^2}{4},\ f_c = \frac{\pi}{4}.$$

It serves as an example of application of the renormalization method when we do not know the critical index and amplitude. This example was used for random composites in [9]. The best formula with known critical index and amplitude was deduced in the book [10] (pp. 55–56).

In this work, we construct the generalization of the MF formula allowing for sufficient extension of intervals of its applicability. Our research is based on the asymptotic homogenization method. Using a multiple scale asymptotic approach, we subdivide the original problem into both a local cell problem and a global problem for the whole composite with homogenized (effective) properties. For approximate analytical solving of the cell problem, we use the Schwarz alternating method (SAM). As a result, we obtain analytical expressions for the effective properties of the two-phase composite material under consideration.

The MF coincides with one of the Hashin–Shtrikman bounds [11]. Namely, the MF equals the Hashin–Shtrikman lower bound if the physical characteristics of the inclusions are larger than that of the matrix, and MF equals the Hashin–Shtrikman upper bound if the physical characteristics of the inclusions are smaller than that of the matrix.

Originally the MF was obtained by solving the problem of conductivity of a dilute suspension of conducting spheres in a conducting matrix. Thus, strictly speaking, the MF was obtained assuming a small volume fraction c of inclusions: $c \ll 1$. But the formula works quite well in the cases of medium and large sizes of inclusions of arbitrary shapes other than high contrast composites; see [5,6]. The boundary perturbation method was applied in [12] for the construction of the first amendment of the MF in the case of square cylindrical inclusions. The effective properties of the fibrous composites with fibers of square cross sections are analyzed in [13,14].

It was shown by Berdichevsky [15] that the MF provides a very good approximation for the effective coefficient of thermal conductivity for composites with cubic lattices, and the corrections for different types of lattices (simple cubic, volume-centered and edge-centered) were obtained. The upper and lower estimates of the effective coefficient of the thermal conductivity of composites with spherical inclusions are also given in [15]. Further, it is shown that the MF undervalues the effective coefficient of the thermal conductivity if the conductivity of the inclusions is larger than the conductivity of the matrix, and overvalues it in the opposite case.

The refinement of the MF is an interesting and important task, and it has attracted the attention of many researchers.

A refinement of the MF was obtained by Milton [5] for 2D and 3D composites on account of the geometrical arrangement of the constituents and their physical characteristics.

In [16], most attention to Maxwell's approach is paid to circular disks where the final formulae can be explicitly written. A series in the contrast parameter for the effective conductivity is truncated and the second and third-order terms are analyzed. It is shown that for macroscopically isotropic composites the second-order term does not depend on the location of inclusions, while the third-order term does.

The Clausius–Mossotti formula was further extended in [17] for non-dilute composites with circular fibers by adding higher order terms to the concentration.

Levin et al. [18] has formulated Maxwell's homogenization scheme in terms of the induced dipole moments of the representative volume element (RVE) of actual composite and properly defined equivalent inclusion. Numerical study shows that the proposed version of Maxwell's scheme enables evaluation of the effective properties of both periodic and random structure composites with accuracy, comparable with that of Rayleigh's method.

The paper [19] is concerned with the problem of the conductivity of double-periodic composite materials with circular inclusions. An exact formula for the tensor of effective conductivity is given, which in the dilute case reduces to the Maxwell formula.

In a number of publications, the Maxwell approach was developed in combination with the cluster method. Thus, in paper [20] extensions of Maxwell's self-consistent approach from single- to n-inclusions problems lead to cluster methods applied to the computation of the effective properties of composites. Paper [21] is devoted to the cluster method for two dimensional elasticity problems and shows how to apply the cluster method to 2D elastic composites.

Analysis of the so far reviewed papers allows us to draw the following conclusions:

1. There is a large number of theoretical and practical problems in the theory of composites, for their solution it is natural to apply Maxwell-type formulas.
2. Unfortunately, the accuracy of Maxwell's formula and its modifications is not always sufficient from the modern point of view.
3. Refinement of Maxwell's formula can be based on some physically justified hypotheses [18], and by constructing the higher-order approximations of some asymptotic processes [7,9,12–14,17,22,23].

The present paper is devoted to the derivation of the refinement of the MF for fiber-reinforced composites with cylindrical fibers of circular cross sections. It is based on the homogenization method with the analytical solution of cell problems using the Schwarz and Padé approximation methods.

The paper is organized as follows. The statement of the problem is described in Sect. 2. The unit-cell problems are formulated and solved in Sect. 3. The refined Maxwell formula is derived in Sect. 4. The analysis of the obtained corrections to the MF is provided in Sect. 5. Numerical results are analyzed in Sect. 6. Finally, Sect. 7 presents the concluding remarks.

2 Statement of the problem

In the case of the heat transfer problem regarding a double-periodic composite structure with a small ($a \ll 1$) cylindrical inclusion of circular cross sections, the local problem (in frame of the averaging method) can be defined in the following way:

$$\frac{\partial^2 u_1^+}{\partial \xi^2} + \frac{\partial^2 u_1^+}{\partial \eta^2} = 0 \quad \text{in } \Omega_i^+; \tag{2.1}$$

$$\frac{\partial^2 u_1^-}{\partial \xi^2} + \frac{\partial^2 u_1^-}{\partial \eta^2} = 0 \quad \text{in } \Omega_i^-; \tag{2.2}$$

$$\begin{gathered} u_1^+ = u_1^-; \quad \frac{\partial u_1^+}{\partial \xi} - \lambda \frac{\partial u_1^-}{\partial \xi} = (\lambda - 1)\frac{\partial u_0}{\partial \xi} \quad \text{for } \xi = \pm a; \\ u_1^+ = u_1^- \quad \frac{\partial u_1^+}{\partial \eta} - \lambda \frac{\partial u_1^-}{\partial \eta} = (\lambda - 1)\frac{\partial u_0}{\partial \eta} \quad \text{for } \eta = \pm a; \end{gathered} \tag{2.3}$$

$$\begin{gathered} u_1^+ \mid_{\xi=1} = u_1^+ \mid_{\xi=-1}; \quad \frac{\partial u_1^+}{\partial \xi} \mid_{\xi=1} = \frac{\partial u_1^+}{\partial \xi} \mid_{\xi=-1}; \quad u_1^+ \mid_{\eta=1} = u_1^+ \mid_{\eta=-1}; \\ \frac{\partial u_1^+}{\partial \eta} \mid_{\eta=1} = \frac{\partial u_1^+}{\partial \eta} \mid_{\eta=-1}. \end{gathered} \tag{2.4}$$

3 Solution of the unit-cell problems

We will use Schwarz alternating method (SAM) [9,22–26] to solve the problem, as it yields a solution of the problem within the (01) approximation of SAM, which refers to the problem formulated for circular inclusions $\Omega_i^{-\,(0)}$ embedded into the infinite space $\Omega_i^{+\,(0)}$. Note that the convergence of the Schwarz algorithm under fairly general assumptions was proved in [27]. Mikhlin [27] stated the problem of convergence and proved it for a doubly connected domain. He also noticed that the method should work for a multiple connected domain when the holes are far away one from another. The convergence of the Schwarz alternating method in the general case for an arbitrary multiply connected domain was proved in publications summarized in Chapter 3 of [9].

In the latter case, the problem defined on the cell (2.1)–(2.4) can be recast in the fast polar coordinates r , θ, and it is governed by the following equations:

$$\frac{\partial^2 u_1^{-(01)}}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u_1^{-(01)}}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u_1^{-(01)}}{\partial \theta^2} = 0 \quad \text{in } \Omega_i^{-(0)}; \tag{3.1}$$

$$\frac{\partial^2 u_1^{+(01)}}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u_1^{+(01)}}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u_1^{+(01)}}{\partial \theta^2} = 0 \quad \text{in } \Omega_i^{+(0)}; \tag{3.2}$$

$$u_1^{+(01)} = u_1^{-(01)};\quad \frac{\partial u_1^{+(01)}}{\partial r} - \lambda\frac{\partial u_1^{-(01)}}{\partial r} = (\lambda-1)\left(\frac{\partial u_0}{\partial x}\cos\theta + \frac{\partial u_0}{\partial y}\sin\theta\right) \quad \text{for } r = \tilde{a}; \tag{3.3}$$

$$u_1^{+(01)} \to 0;\quad \frac{\partial u_1^{+(01)}}{\partial r} \to 0 \quad \text{for } r \to \infty. \tag{3.4}$$

A solution to the coupled problem (3.1)–(3.4) takes the form

$$u_1^{-\,(01)} = A_1^{(01)}\, r\cos\,\theta + A_2^{(01)}\, r\sin\,\theta, \tag{3.5}$$

$$u_1^{+\,(01)} = \frac{B_1^{(01)}}{r}\,\cos\,\theta + \frac{B_2^{(01)}}{r}\,\sin\,\theta, \tag{3.6}$$

where $A_1^{(01)}$, $A_2^{(01)}$, $B_1^{(01)}$, $B_2^{(01)}$ are arbitrary constants.

It is worthy to mention that representation of the function $u_1^{-\,(01)}$ in the form of (3.5) includes the boundaries of the function governing temperature distribution and its derivation $\frac{\partial u_1^{-\,(01)}}{\partial r}$ (heat stream in radial direction) for $r = 0$, whereas the function $u_1^{+\,(01)}$ (3.6) satisfies conditions of damping of the latter characteristics for $r \to \infty$ (3.4).

Observe that relations (3.5), (3.6) include four arbitrary constants, i.e. each two for them stand as the basic functions cos θ and sin θ ($A_1^{(01)}$, $B_1^{(01)}$ and $A_2^{(01)}$, $B_2^{(01)}$, respectively), which are defined through compatibility conditions (3.3). Since the systems of equations which require definition of the integration constants $A_1^{(01)}$, $B_1^{(01)}$ and $A_2^{(01)}$, $B_2^{(01)}$ are the same to avoid repetitions, to avoid repetition, we present only one of them:

$$\begin{cases} A_1^{(01)}\, a = B_1^{(01)}\, a^{-1}, \\ -B_1^{(01)}\, a^{-2} - \lambda A_1^{(01)} = \frac{\partial u_0}{\partial x}\,(\lambda-1)\,. \end{cases} \tag{3.7}$$

Solving equations (3.7) allows to find the integration constants

$$\begin{cases} A_1^{(01)} = -\frac{\lambda-1}{\lambda+1}\,\frac{\partial u_0}{\partial x} = \frac{\partial u_0}{\partial x}\,A^{(01)\,*}, \\ B_1^{(01)} = -\frac{(\lambda-1)\,a^2}{\lambda+1}\,\frac{\partial u_0}{\partial x} = \frac{\partial u_0}{\partial x}\,B^{(01)\,*}, \end{cases} \tag{3.8}$$

where

$$A^{(01)\,*} = -\frac{\lambda-1}{\lambda+1} \quad \text{and} \quad B^{(01)\,*} = -\frac{(\lambda-1)\,a^2}{\lambda+1}. \tag{3.9}$$

It is clear that for arbitrary constants $A_2^{(01)}$ and $B_2^{(01)}$ we have

$$A_2^{(01)} = A_1^{(01)} \quad \text{and} \quad B_2^{(01)} = B_1^{(01)}\,\left(\frac{\partial u_0}{\partial x} \to \frac{\partial u_0}{\partial y}\right). \tag{3.10}$$

Therefore, the solution of the (01) approximation is as follows:

$$u_1^{-(01)}=-\frac{\partial u_0}{\partial x}\frac{\lambda-1}{\lambda+1}r\cos\theta-\frac{\partial u_0}{\partial y}\frac{\lambda-1}{\lambda+1}r\sin\theta,\quad u_1^{+(01)}=-\frac{\partial u_0}{\partial x}\frac{\lambda-1}{\lambda+1}a^2\frac{\cos\theta}{r}-\frac{\partial u_0}{\partial y}\frac{\lambda-1}{\lambda+1}a^2\frac{\sin\theta}{r}, \tag{3.11}$$

or equivalently

$$u_1^{-(01)}=-\frac{\lambda-1}{\lambda+1}\left(\frac{\partial u_0}{\partial x}\xi+\frac{\partial u_0}{\partial y}\eta\right),\quad u_1^{+(01)}=-\frac{\partial u_0}{\partial x}\frac{\lambda-1}{\lambda+1}a^2\left(\frac{\partial u_0}{\partial x}\frac{\xi}{\xi^2+\eta^2}+\frac{\partial u_0}{\partial y}\frac{\eta}{\xi^2+\eta^2}\right). \tag{3.11'}$$

In what follows, we construct the (02) approximation of the SAM, which refers to the solution of the problem in the cell matrix Ω_i^*.

Now, the periodicity conditions (2.4) located on opposite sides of the cell are satisfied, and the compatibility conditions (2.3) are ignored. Since the function $u_1^{(02)}$ should correct errors occurring in the solution $u_1^{+\,(01)}$ on the sizes of the cell, the following boundary value problem holds:

$$\Delta u_1^{(02)}=0\quad \text{in}\ \Omega_i^*; \tag{3.12}$$

$$\begin{gathered}\left(u_1^{+\,(01)}+u_1^{(02)}\right)\Big|_{\xi=1}=\left(u_1^{+\,(01)}+u_1^{(02)}\right)\Big|_{\xi=-1},\\ \frac{\partial\left(u_1^{+\,(01)}+u_1^{(02)}\right)}{\partial\xi}\Bigg|_{\xi=1}=\frac{\partial\left(u_1^{+\,(01)}+u_1^{(02)}\right)}{\partial\xi}\Bigg|_{\xi=-1},\end{gathered} \tag{3.13}$$

$$\begin{gathered}\left(u_1^{+\,(01)}+u_1^{(02)}\right)\Big|_{\eta=1}=\left(u_1^{+\,(01)}+u_1^{(02)}\right)\Big|_{\eta=-1},\\ \frac{\partial\left(u_1^{+\,(01)}+u_1^{(02)}\right)}{\partial\eta}\Bigg|_{\eta=1}=\frac{\partial\left(u_1^{+\,(01)}+u_1^{(02)}\right)}{\partial\eta}\Bigg|_{\eta=-1}.\end{gathered} \tag{3.14}$$

We assume

$$u_1^{(02)}=u_{11}^{(02)}+u_{12}^{(02)}, \tag{3.15}$$

where $u_{11}^{(02)}$ satisfies non-homogenous boundary conditions with regard to ξ and homogenous boundary condition with regard to η. Therefore, the following equations should be satisfied:

$$\Delta u_{11}^{(02)}=0\quad \text{in}\ \Omega_i^*; \tag{3.16}$$

$$\begin{gathered}\left(u_1^{+\,(01)}+u_{11}^{(02)}\right)\Big|_{\xi=1}=\left(u_1^{+\,(01)}+u_{11}^{(02)}\right)\Big|_{\xi=-1},\\ \frac{\partial\left(u_1^{+\,(01)}+u_{11}^{(02)}\right)}{\partial\xi}\Bigg|_{\xi=1}=\frac{\partial\left(u_1^{+\,(01)}+u_{11}^{(02)}\right)}{\partial\xi}\Bigg|_{\xi=-1},\end{gathered} \tag{3.17}$$

$$u_{11}^{(02)}\Big|_{\eta=1}=u_{11}^{(02)}\Big|_{\eta=-1};\quad \frac{\partial u_{11}^{(02)}}{\partial\eta}\Bigg|_{\eta=1}=\frac{\partial u_{11}^{(02)}}{\partial\eta}\Bigg|_{\eta=-1}. \tag{3.18}$$

It is obvious that the function $u_{12}^{(02)}$ can be found in an analogous way simply by using the change: $\xi \rightleftarrows \eta$.

A general solution of Eq. (3.16) takes the form

$$\begin{aligned}u_{11}^{(02)}&=A_0^{(02)}+B_0^{(02)}\xi+\sum_{n=1}^{\infty}\Big[\left(A_\mathrm{n}^{(02)}\cosh\pi n\xi+B_\mathrm{n}^{(02)}\sinh\pi n\xi\right)\cos\pi n\eta\\&\quad+\left(C_\mathrm{n}^{(02)}\cosh\pi n\xi+D_\mathrm{n}^{(02)}\sinh\pi n\xi\right)\sin\pi n\eta\Big],\end{aligned} \tag{3.19}$$

where $A_0^{(02)},\ B_0^{(02)},\ A_n^{(02)},\ B_n^{(02)},\ C_n^{(02)},\ D_n^{(02)}\ (n=1,\ 2,\ \ldots)$ are arbitrary constants.

In order to satisfy the boundary conditions (3.17), we recast them considering the (01) approximation of (3.11) to the following form:

$$\begin{aligned} &u_{11}^{(02)}\Big|_{\xi=1} - u_{11}^{(02)}\Big|_{\xi=-1} = \frac{\partial u_0}{\partial x}\cdot\frac{2a^2}{1+\eta^2}\,\frac{\lambda-1}{\lambda+1},\\ &\frac{\partial u_{11}^{(02)}}{\partial\xi}\Big|_{\xi=1} - \frac{\partial u_{11}^{(02)}}{\partial\xi}\Big|_{\xi=-1} = -\frac{\partial u_0}{\partial y}\cdot\frac{4a^2\eta}{(1+\eta^2)^2}\,\frac{\lambda-1}{\lambda+1}. \end{aligned} \tag{3.20}$$

The right-hand sides of equations (3.20) are expand into the following Fourier series:

$$\begin{aligned} &\frac{1}{1+\eta^2} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[e^{-\pi n}\,\mathrm{Im}\,E_1(-\pi n + i\pi n) - e^{\pi n}\,\mathrm{Im}\,E_1(\pi n + i\pi n) + \pi e^{-\pi n}\right]\cos\pi n\eta,\\ &\frac{\eta}{(1+\eta^2)^2} = \frac{\pi}{2}\sum_{n=1}^{\infty} n\left[e^{\pi n}\,\mathrm{Im}\,E_1(\pi n + i\pi n) - e^{-\pi n}\,\mathrm{Im}\,E_1(-\pi n + i\pi n) - \pi e^{-\pi n}\right]\sin\pi n\eta, \end{aligned} \tag{3.21}$$

where $i=\sqrt{-1}$ and E_1 stands for the exponential integral [28].

Comparison of the corresponding coefficients in (3.20), accounting for (3.21) yields the (02) approximation coefficients:

$$A_0^{(02)} = 0;\ A_n^{(02)} = D_n^{(02)} = 0,\quad n = 1,\ 2,\ldots;$$

$$B_0^{(02)} = \frac{\partial u_0}{\partial x}B_0^{(02)*};\quad B_n^{(02)} = \frac{\partial u_0}{\partial x}B_n^{(02)*};\quad C_n^{(02)} = \frac{\partial u_0}{\partial y}C_n^{(02)*},\quad n = 1,\ 2,\ \ldots; \tag{3.22}$$

$$B_0^{(02)*} = \frac{\lambda-1}{\lambda+1}\cdot\frac{\pi a^2}{4}; \tag{3.23}$$

$$B_n^{(02)*} = -C_n^{(02)*} = \frac{\lambda-1}{\lambda+1}a^2 S_n, \tag{3.24}$$

where

$$S_{\mathrm{n}} = \frac{e^{-\pi n}\,\mathrm{Im}\,E_1(-\pi n + i\pi n) - e^{\pi n}\,\mathrm{Im}\,E_1(\pi n + \mathrm{i}\pi n) + \pi e^{-\pi n}}{\sinh\pi n}. \tag{3.25}$$

Consequently, we get

$$u_{11}^{(02)} = \frac{\partial u_0}{\partial x}B_0^{(02)*}\xi + \sum_{n=1}^{\infty}\left(\frac{\partial u_0}{\partial x}B_{\mathrm{n}}^{(02)*}\sinh\pi n\xi\,\cos\pi n\eta - \frac{\partial u_0}{\partial y}C_{\mathrm{n}}^{(02)*}\cosh\pi n\xi\,\sin\pi n\eta\right). \tag{3.26}$$

Proceeding in an analogous way yields

$$u_{12}^{(02)} = u_{11}^{(02)}\left(\frac{\partial u_0}{\partial x}\rightleftarrows\frac{\partial u_0}{\partial y};\ \xi\rightleftarrows\eta\right). \tag{3.27}$$

Finally, the (02) order approximation takes the following form:

$$\begin{aligned} u_1^{(02)} = \frac{\lambda-1}{\lambda+1}\cdot\frac{\pi a^2}{4}\left(\frac{\partial u_0}{\partial x}\xi + \frac{\partial u_0}{\partial y}\eta\right) + \frac{\lambda-1}{\lambda+1}a^2\sum_{n=1}^{\infty}S_{\mathrm{n}}\Bigg[\frac{\partial u_0}{\partial x}(\sinh\pi n\xi\,\cos\pi n\eta -\\ -\cosh\pi n\eta\,\sin\pi n\xi) + \frac{\partial u_0}{\partial y}(\sinh\pi n\eta\,\cos\pi n\xi - \cosh\pi n\xi\,\sin\pi n\eta)\Bigg]. \end{aligned} \tag{3.28}$$

In the (03) order approximation, we should remove the lack of compliance of the function $u_1^{(02)}$ governed by (3.28), on the circular contour of an inclusion with radius $r=a$. For this purpose, we develop the function $u_1^{(02)}$ into a series regarding polar coordinates r, θ; assuming a small radius r of the inclusions, we obtain

$$\begin{aligned} B_0^{(02)*}\left(\frac{\partial u_0}{\partial x}\xi + \frac{\partial u_0}{\partial y}\eta\right) + \sum_{n=1}^{\infty}B_{\mathrm{n}}^{(02)*}\Bigg[\frac{\partial u_0}{\partial x}(\sinh\pi n\xi\,\cos\pi n\eta - \cosh\pi n\eta\,\sin\pi n\xi)\\ +\frac{\partial u_0}{\partial y}(\sinh\pi n\eta\,\cos\pi n\xi - \cosh\pi n\xi\,\sin\pi n\eta)\Bigg] = B_0^{(02)*}\left(\frac{\partial u_0}{\partial x}r\cos\theta + \frac{\partial u_0}{\partial y}r\sin\theta\right) \end{aligned}$$

$$+2\sum_{n=1}^{\infty} B_n^{(02)\,*}\left[\frac{\partial u_0}{\partial x}\sum_{k=1}^{\infty}\frac{(\pi n r)^{4k-1}\cos\,(4k-1)\,\theta}{(4k-1)\,!}+\frac{\partial u_0}{\partial y}\sum_{k=1}^{\infty}\frac{(\pi n r)^{4k-1}\sin\,(4k-1)\,\theta}{(4k-1)\,!}\right],$$

or

$$B_0^{(02)\,*}\left(\frac{\partial u_0}{\partial x}\,\xi+\frac{\partial u_0}{\partial y}\,\eta\right)+\sum_{n=1}^{\infty} B_n^{(02)\,*}\left[\frac{\partial u_0}{\partial x}\,(\sinh\,\pi n\xi\,\cos\,\pi n\eta-\cosh\,\pi n\eta\,\sin\,\pi n\xi)\right.$$
$$\left.+\frac{\partial u_0}{\partial y}\,(\sinh\,\pi n\eta\,\cos\,\pi n\xi-\cosh\,\pi n\xi\,\sin\,\pi n\eta)\right]=B_0^{(02)\,*}\left(\frac{\partial u_0}{\partial x}\,r\cos\,\theta+\frac{\partial u_0}{\partial y}\,r\sin\,\theta\right)$$
$$+2\sum_{k=1}^{\infty}\frac{\pi^{4k-1}}{(4k-1)\,!}\left(\sum_{n=1}^{\infty}B_n^{(02)\,*}n^{4k-1}\right)\left(\frac{\partial u_0}{\partial x}r^{4k-1}\cos\,(4k-1)\,\theta\right.$$
$$\left.+\frac{\partial u_0}{\partial y}r^{4k-1}\sin\,(4k-1)\,\theta\right). \tag{3.29}$$

Observe that now the right-hand side of (3.29) is convergent for all values of $0\le r<\infty$.

The correcting terms of the (03) approximation follow:

$$u_1^{-\,(03)}=A_{10}^{(03)}\,r\cos\,\theta+A_{20}^{(03)}\,r\sin\,\theta$$
$$+\sum_{k=1}^{\infty}\left(A_{1k}^{(03)}\,r^{4k-1}\cos\,(4k-1)\,\theta+A_{2k}^{(03)}\,r^{4k-1}\sin\,(4k-1)\,\theta\right); \tag{3.30}$$

$$u_1^{+\,(03)}=\frac{B_{10}^{(03)}}{r}\cos\,\theta+\frac{B_{20}^{(03)}}{r}\sin\,\theta+\sum_{k=1}^{\infty}\left(B_{1k}^{(03)}\frac{\cos\,(4k-1)\,\theta}{r^{4k-1}}+B_{2k}^{(03)}\frac{\sin\,(4k-1)\,\theta}{r^{4k-1}}\right), \tag{3.31}$$

where the constants

$$A_{10}^{(03)}=A_{10}^{(03)*}\frac{\partial u_0}{\partial x},\quad B_{10}^{(03)}=B_{10}^{(03)*}\frac{\partial u_0}{\partial x};$$
$$A_{20}^{(03)}=A_{20}^{(03)*}\frac{\partial u_0}{\partial y},\quad B_{20}^{(03)}=B_{20}^{(03)*}\frac{\partial u_0}{\partial y}$$

and

$$A_{1k}^{(03.n)}=A_{1k}^{(03.n)*}\frac{\partial u_0}{\partial x},\quad B_{1k}^{(03.n)}=B_{1k}^{(03.n)*}\frac{\partial u_0}{\partial x};$$
$$A_{2k}^{(03.n)}=A_{2k}^{(03.n)*}\frac{\partial u_0}{\partial y},\quad B_{2k}^{(03.n)}=B_{2k}^{(03.n)*}\frac{\partial u_0}{\partial y}$$

are defined through solution of either the following

$$\begin{cases} A_{m0}^{(03)*}\,a=\frac{B_{m0}^{(03)*}}{a}\\ -\frac{B_{m0}^{(03)*}}{a^2}-\lambda\,A_{m0}^{(03)*}=(\lambda-1)\;B_0^{(02)*}\end{cases}$$

or the following

$$\begin{cases} A_{mk}^{(03.n)*}\,a^{4k-1}=\frac{B_{mk}^{(03.n)*}}{a^{4k-1}}\\ -\frac{B_{mk}^{(03.n)*}}{\tilde a^{4k}}-\lambda\,a^{4k-2}A_{mk}^{(03.n)*}=\frac{2\,(\lambda-1)\pi^{4k-1}a^{4k-2}}{(4k-1)\,!}\sum_{n=1}^{\infty}B_n^{(02)\,*}n^{4k-1}\end{cases} \tag{3.32}$$

system of equations. Therefore, we get

$$A_{m0}^{(03)\,*}=-\frac{\lambda-1}{\lambda+1}\,B_0^{(02)\,*}=-\left(\frac{\lambda-1}{\lambda+1}\right)^2\frac{\pi a^2}{4},\quad B_{m0}^{(03)\,*}=-\frac{\lambda-1}{\lambda+1}\,a^2B_0^{(02)\,*}=-\left(\frac{\lambda-1}{\lambda+1}\right)^2\frac{\pi a^4}{4},$$

$$A_{mk}^{(03)*} = -\frac{2\,\pi^{4k-1}}{(4k-1)\,!} \cdot \frac{\lambda-1}{\lambda+1} \sum_{n=1}^{\infty} B_n^{(02)\,*} n^{4k-1} = -\frac{2\,\pi^{4k-1} a^2}{(4k-1)\,!} \cdot \left(\frac{\lambda-1}{\lambda+1}\right)^2 \sum_{n=1}^{\infty} S_n n^{4k-1}, \tag{3.33}$$

$$B_{mk}^{(03)*} = -\frac{2\,\pi^{4k-1} a^{8k-2}}{(4k-1)\,!} \cdot \frac{\lambda-1}{\lambda+1} \sum_{n=1}^{\infty} B_n^{(02)\,*} n^{4k-1} = -\frac{2\,\pi^{4k-1} a^{8k}}{(4k-1)\,!} \cdot \left(\frac{\lambda-1}{\lambda+1}\right)^2 \sum_{n=1}^{\infty} S_n n^{4k-1}. \tag{3.34}$$

In other words, the final form of the (03) approximation governed by (3.30), (3.31), accounting for (3.33), (3.34), is written in the following form:

$$\begin{aligned} u_1^{-\,(03)} &= -\frac{\lambda-1}{\lambda+1} B_0^{(02)\,*} r \left(\frac{\partial u_0}{\partial x} \cos\,\theta + \frac{\partial u_0}{\partial y} \sin\,\theta\right) \\ &-\frac{\lambda-1}{\lambda+1} 2 \left(\sum_{n=1}^{\infty} B_n^{(02)\,*} n^{4k-1}\right) \sum_{k=1}^{\infty} \frac{\pi^{4k-1} r^{4k-1}}{(4k-1)\,!} \left(\frac{\partial u_0}{\partial x} \cos\,(4k-1)\,\theta + \frac{\partial u_0}{\partial y} \sin\,(4k-1)\,\theta\right); \end{aligned} \tag{3.35}$$

$$\begin{aligned} u_1^{+\,(03)} &= -\frac{\lambda-1}{\lambda+1} \frac{B_0^{(02)\,*}}{r} \left(\frac{\partial u_0}{\partial x} \cos\,\theta + \frac{\partial u_0}{\partial y} \sin\,\theta\right) \\ &-\frac{\lambda-1}{\lambda+1} 2 \left(\sum_{n=1}^{\infty} B_n^{(02)\,*} n^{4k-1}\right) \sum_{k=1}^{\infty} \frac{\pi^{4k-1} a^{8k-2}}{(4k-1)\,!\, r^{4k-1}} \left(\frac{\partial u_0}{\partial x} \cos\,(4k-1)\,\theta \right.\\ &\left.+\frac{\partial u_0}{\partial y} \sin\,(4k-1)\,\theta\right). \end{aligned} \tag{3.36}$$

Reversing the numbering of the series due to k in formulas (3.35), (3.36) yields

$$\begin{aligned} u_1^{-\,(03)} &= -\frac{\lambda-1}{\lambda+1} B_0^{(02)*} \left(\frac{\partial u_0}{\partial x}\xi + \frac{\partial u_0}{\partial y}\eta\right) - \frac{\lambda-1}{\lambda+1} \sum_{n=1}^{\infty} B_{\mathrm{n}}^{(02)*} \left[\frac{\partial u_0}{\partial x} (\sinh\,\pi n\xi\,\cos\,\pi n\eta \right. \\ &\left. -\cosh\,\pi n\eta\,\sin\,\pi n\xi) + \frac{\partial u_0}{\partial y} (\sinh\,\pi n\eta\,\cos\,\pi n\xi - \cosh\,\pi n\xi\,\sin\,\pi n\eta)\right]; \end{aligned} \tag{3.37}$$

$$\begin{aligned} u_1^{+\,(03)} &= -\left(\frac{\lambda-1}{\lambda+1}\right)^2 \frac{\pi a^2}{4} \left(\frac{\partial u_0}{\partial x} \frac{\xi}{\xi^2+\eta^2} + \frac{\partial u_0}{\partial y} \frac{\eta}{\xi^2+\eta^2}\right) \\ &-\frac{\lambda-1}{\lambda+1} \sum_{n=1}^{\infty} B_{\mathrm{n}}^{(02)*} \left[\frac{\partial u_0}{\partial x} \left(\sinh\,\frac{\pi n a^2 \xi}{\xi^2+\eta^2}\,\cos\,\frac{\pi n a^2 \eta}{\xi^2+\eta^2} - \cosh\,\frac{\pi n a^2 \eta}{\xi^2+\eta^2}\,\sin\,\frac{\pi n a^2 \xi}{\xi^2+\eta^2}\right)\right. \\ &\left.+\frac{\partial u_0}{\partial y} \left(\sinh\,\frac{\pi n a^2 \eta}{\xi^2+\eta^2}\,\cos\,\frac{\pi n a^2 \xi}{\xi^2+\eta^2} - \cosh\,\frac{\pi n a^2 \xi}{\xi^2+\eta^2}\,\sin\,\frac{\pi n a^2 \eta}{\xi^2+\eta^2}\right)\right], \end{aligned} \tag{3.38}$$

i.e., we have

$$\begin{aligned} u_1^{-\,(03)} &= -\left(\frac{\lambda-1}{\lambda+1}\right)^2 \frac{\pi a^2}{4} \left(\frac{\partial u_0}{\partial x}\xi + \frac{\partial u_0}{\partial y}\eta\right) - \left(\frac{\lambda-1}{\lambda+1}\right)^2 a^2 \sum_{n=1}^{\infty} S_{\mathrm{n}} \left[\frac{\partial u_0}{\partial x} (\sinh\,\pi n\xi\,\cos\,\pi n\eta \right. \\ &\left. -\cosh\,\pi n\eta\,\sin\,\pi n\xi) + \frac{\partial u_0}{\partial y} (\sinh\,\pi n\eta\,\cos\,\pi n\xi - \cosh\,\pi n\xi\,\sin\,\pi n\eta)\right]; \end{aligned} \tag{3.39}$$

$$\begin{aligned} u_1^{+\,(03)} &= -\left(\frac{\lambda-1}{\lambda+1}\right)^2 \frac{\pi a^2}{4} \left(\frac{\partial u_0}{\partial x} \frac{\xi}{\xi^2+\eta^2} + \frac{\partial u_0}{\partial y} \frac{\eta}{\xi^2+\eta^2}\right) \\ &-\left(\frac{\lambda-1}{\lambda+1}\right)^2 a^2 \sum_{n=1}^{\infty} S_{\mathrm{n}} \left[\frac{\partial u_0}{\partial x} \left(\sinh\,\frac{\pi n a^2 \xi}{\xi^2+\eta^2}\,\cos\,\frac{\pi n a^2 \eta}{\xi^2+\eta^2} - \cosh\,\frac{\pi n a^2 \eta}{\xi^2+\eta^2}\,\sin\,\frac{\pi n a^2 \xi}{\xi^2+\eta^2}\right)\right. \\ &\left.+\frac{\partial u_0}{\partial y} \left(\sinh\,\frac{\pi n a^2 \eta}{\xi^2+\eta^2}\,\cos\,\frac{\pi n a^2 \xi}{\xi^2+\eta^2} - \cosh\,\frac{\pi n a^2 \xi}{\xi^2+\eta^2}\,\sin\,\frac{\pi n a^2 \eta}{\xi^2+\eta^2}\right)\right]. \end{aligned} \tag{3.40}$$

In the (04) approximation, we proceed in analogous way to remove lack of compliance of the function $u_1^{+\,(03)}$ of (3.40) on the external contour of the matrix.

We define for the function

$$u_{11}^{(04)} = A_0^{(04)} + B_0^{(04)}\,\xi + \sum_{\ell=1}^{\infty}\Big[\Big(A_\ell^{(04)}\cosh\pi\ell\xi + B_\ell^{(04)}\sinh\pi\ell\xi\Big)\cos\pi\ell\eta + \Big(C_\ell^{(02)}\cosh\pi\ell\xi + D_\ell^{(02)}\sinh\pi\ell\xi\Big)\sin\pi\ell\eta\Big]$$

the following boundary conditions for the cell at $\xi = 1$ and $\xi = -1$:

$$\begin{aligned} u_{11}^{(04)}\,\big|_{\xi=1} - u_{11}^{(04)}\,\big|_{\xi=-1} &= \frac{\partial u_0}{\partial x}\frac{\lambda-1}{\lambda+1}B_0^{(02)*}\frac{2}{1+\eta^2} \\ &+\frac{\partial u_0}{\partial x}\frac{\lambda-1}{\lambda+1}2\sum_{n=1}^{\infty}B_n^{(02)*}\left(\sinh\frac{\pi na^2}{1+\eta^2}\cos\frac{\pi na^2\eta}{1+\eta^2} - \cosh\frac{\pi na^2\eta}{1+\eta^2}\sin\frac{\pi na^2}{1+\eta^2}\right); \end{aligned} \tag{3.41}$$

$$\begin{aligned} \frac{\partial u_{11}^{(04)}}{\partial\xi}\,\Big|_{\xi=1} - \frac{\partial u_{11}^{(04)}}{\partial\xi}\,\Big|_{\xi=-1} &= -\frac{\partial u_0}{\partial y}\frac{\lambda-1}{\lambda+1}B_0^{(02)*}\frac{4\eta}{(1+\eta^2)^2} \\ &-\frac{\partial u_0}{\partial y}\frac{\lambda-1}{\lambda+1}2\pi a^2\sum_{n=1}^{\infty}B_n^{(02)*}n\left[\frac{2\eta}{(1+\eta^2)^2}\cosh\frac{\pi na^2\eta}{1+\eta^2}\cos\frac{\pi na^2}{1+\eta^2}\right. \\ &-\frac{1-\eta^2}{(1+\eta^2)^2}\sinh\frac{\pi na^2\eta}{1+\eta^2}\sin\frac{\pi na^2}{1+\eta^2} - \frac{1-\eta^2}{(1+\eta^2)^2}\sinh\frac{\pi na^2}{1+\eta^2}\sin\frac{\pi na^2\eta}{1+\eta^2} \\ &\left.-\frac{2\eta}{(1+\eta^2)^2}\cosh\frac{\pi na^2}{1+\eta^2}\cos\frac{\pi na^2\eta}{1+\eta^2}\right]. \end{aligned} \tag{3.42}$$

Assuming $a \ll 1$ and developing the functions standing on the right-hand sides of the formulas (3.41), (3.42) into the Fourier series, allows to define the following coefficients:

$$A_0^{(04)} = 0;\quad A_\ell^{(04)} = D_\ell^{(04)} = 0; \tag{3.43}$$

$$B_0^{(04)} = \frac{\partial u_0}{\partial x}B_0^{(04)*};\quad B_\ell^{(04)} = \frac{\partial u_0}{\partial x}B_\ell^{(04)*};\quad C_\ell^{(04)} = \frac{\partial u_0}{\partial y}C_\ell^{(04)*},\quad \ell = 0,\ 1,\ 2,\ \ldots; \tag{3.44}$$

$$\begin{aligned} B_0^{(04)*} &= \frac{\lambda-1}{\lambda+1}\frac{\pi a^2}{4}B_0^{(02)*} + \frac{\lambda-1}{\lambda+1}\sum_{n=1}^{\infty}B_n^{(02)*}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\left(\pi na^2\right)^{4m-1}}{2^{2m-1}\,(2m-1)\,(4m-1)\,!} \\ &= \left(\frac{\lambda-1}{\lambda+1}\right)^2\frac{\pi^2a^4}{16}\left(1+\frac{8}{\pi}\sum_{n=1}^{\infty}S_n n\sum_{m=1}^{\infty}\frac{(-1)^{m+1}(\pi n)^{4m-2}a^{8m-4}}{2^{2m-2}\,(2m-1)\,(4m-1)\,!}\right); \end{aligned} \tag{3.45}$$

$$\begin{aligned} B_\ell^{(04)*} = -C_\ell^{(04)*} &= \frac{\lambda-1}{\lambda+1}a^2B_0^{(02)*}S_\ell + \frac{\lambda-1}{\lambda+1}\sum_{n=1}^{\infty}B_n^{(02)*}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\left(\pi na^2\right)^{4m-1}}{(2m-1)\,(4m-1)\,!\,(4m-3)\,!} \\ &\times\left((\pi\ell)^{4m-2}S_\ell + \sum_{k=1}^{m}(-1)^{\ell+k+1}\frac{(4k-3)\,!}{2^{2k-2}}\frac{(\pi\ell)^{4m-4k}}{\sinh\pi\ell}\right) \\ &= \left(\frac{\lambda-1}{\lambda+1}\right)^2\frac{\pi a^4}{4}\left[S_\ell + 4\sum_{n=1}^{\infty}S_n n\sum_{m=1}^{\infty}\frac{(-1)^{m+1}(\pi n)^{4m-2}a^{8m-4}}{(2m-1)\,(4m-1)\,!\,(4m-3)\,!}\right. \\ &\left.\times\left((\pi\ell)^{4m-2}S_\ell + \sum_{k=1}^{m}(-1)^{\ell+k+1}\frac{(4k-3)\,!}{2^{2k-2}}\frac{(\pi\ell)^{4m-4k}}{\sinh\pi\ell}\right)\right]. \end{aligned} \tag{3.46}$$

Consequently, the (04) approximation takes the following form:

$$
\begin{aligned}
u_1^{(04)} &= \left(\frac{\lambda-1}{\lambda+1}\right)^2 \frac{\pi^2 a^4}{16}\left(1+\frac{8}{\pi}\sum_{n=1}^{\infty} S_n n \sum_{m=1}^{\infty}\frac{(-1)^{m+1}(\pi n)^{4m-2}\, a^{8m-4}}{2^{2m-2}(2m-1)\,(4m-1)\,!}\right)\left(\frac{\partial u_0}{\partial x}\,\xi+\frac{\partial u_0}{\partial y}\,\eta\right)\\
&+\left(\frac{\lambda-1}{\lambda+1}\right)^2 \frac{\pi a^4}{4}\left[\sum_{\ell=1}^{\infty} S_\ell + 4\sum_{\ell=1}^{\infty} S_\ell \sum_{n=1}^{\infty} S_n n \sum_{m=1}^{\infty}\frac{(-1)^{m+1}\left(\pi^2 n\ell\right)^{4m-2} a^{8m-4}}{(2m-1)\,(4m-1)\,!\,(4m-3)\,!}\right.\\
&\left.+4\sum_{\ell=1}^{\infty}\frac{1}{\sinh\,\pi\ell}\sum_{\mathrm{n}=1}^{\infty} S_{\mathrm{n}} n\sum_{\mathrm{m}=1}^{\infty}\frac{(-1)^{\mathrm{m}+1}(\pi n)^{4\mathrm{m}-2}\, a^{8\mathrm{m}-4}}{(2m-1)\,(4m-1)\,!\,(4m-3)}\,!\sum_{k=1}^{\mathrm{m}}\frac{(-1)^{\ell+k+1}\,(4k-3)\,!\,(\pi\ell)^{4\mathrm{m}-4\mathrm{k}}}{2^{2\mathrm{k}-2}}\right]\\
&\times\left[\frac{\partial u_0}{\partial x}\,(\sinh\,\pi\ell\xi\,\cos\,\pi\ell\eta-\cosh\,\pi\ell\eta\,\sin\,\pi\ell\xi)+\frac{\partial u_0}{\partial y}\,(\sinh\,\pi\ell\eta\,\cos\,\pi\ell\xi-\cosh\,\pi\ell\xi\,\sin\,\pi\ell\eta)\right].
\end{aligned}
\tag{3.47}
$$

4 The refined Maxwell formula

A further extension of an iteration process can be carried out in an analogous way, and an averaging procedure of the following equations

$$
\begin{aligned}
&\left(1-a^2+\lambda a^2\right)\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}\right)+\frac{1}{|\Omega_i^*|}\left[\iint\limits_{\Omega_i^+}\left(\frac{\partial^2 u_1^{+\,(01)}}{\partial x\partial\xi}+\frac{\partial^2 u_1^{+\,(01)}}{\partial y\partial\eta}+\frac{\partial^2 u_1^{(02)}}{\partial x\partial\xi}+\frac{\partial^2 u_1^{(02)}}{\partial y\partial\eta}\right.\right.\\
&+\frac{\partial^2 u_1^{+\,(03)}}{\partial x\partial\xi}+\frac{\partial^2 u_1^{+\,(03)}}{\partial y\partial\eta}+\frac{\partial^2 u_1^{(04)}}{\partial x\partial\xi}+\frac{\partial^2 u_1^{(04)}}{\partial y\partial\eta}+\frac{\partial^2 u_1^{+\,(05)}}{\partial x\partial\xi}+\frac{\partial^2 u_1^{+\,(05)}}{\partial y\partial\eta}\\
&\left.+\frac{\partial^2 u_1^{(06)}}{\partial x\partial\xi}+\frac{\partial^2 u_1^{(06)}}{\partial y\partial\eta}+\ldots\right)\mathrm{d}\xi\,\mathrm{d}\eta+\lambda\iint\limits_{\Omega_i^-}\left(\frac{\partial^2 u_1^{-\,(01)}}{\partial x\partial\xi}+\frac{\partial^2 u_1^{-\,(01)}}{\partial y\partial\eta}+\frac{\partial^2 u_1^{(02)}}{\partial x\partial\xi}+\frac{\partial^2 u_1^{(02)}}{\partial y\partial\eta}\right.\\
&+\frac{\partial^2 u_1^{-\,(03)}}{\partial x\partial\xi}+\frac{\partial^2 u_1^{-\,(03)}}{\partial y\partial\eta}+\frac{\partial^2 u_1^{(04)}}{\partial x\partial\xi}+\frac{\partial^2 u_1^{(04)}}{\partial y\partial\eta}+\frac{\partial^2 u_1^{-\,(05)}}{\partial x\partial\xi}+\frac{\partial^2 u_1^{-\,(05)}}{\partial y\partial\eta}\\
&\left.\left.+\frac{\partial^2 u_1^{(06)}}{\partial x\partial\xi}+\frac{\partial^2 u_1^{(06)}}{\partial y\partial\eta}+\ldots\right)\mathrm{d}\xi\,\mathrm{d}\eta\right]=F
\end{aligned}
\tag{4.1}
$$

yields the following effective coefficients of the thermal conductivity:

$$
\begin{aligned}
q &= 1+2\,\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}+2\left(\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}\right)^2+2\left(\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}\right)^3+\ldots\\
&+\left(\frac{\lambda-1}{\lambda+1}\right)^3\frac{\pi^2 a^6}{4}\left(1+\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}+\ldots\right)^2\sum_{\ell=1}^{\infty} S_\ell \ell\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(\pi\ell)^{4n-2}\, a^{8n-4}}{2^{2n-1}(2n-1)\,(4n-1)\,!}\\
&\times\left(\sum_{\ell=1}^{\infty} S_\ell \ell+\frac{\pi a^2}{8}\sum_{\ell=1}^{\infty} S_\ell \ell\sum_{j=1}^{\infty} S_j j\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\left(\pi^2\ell\, j\right)^{4m-2} a^{8m-4}}{(2m-1)\,(4m-1)\,!\,(4m-3)\,!}\right.\\
&+\left(\frac{\lambda-1}{\lambda+1}\right)^4\frac{\pi^3 a^8}{16}\left(1+\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}+\ldots\right)^2\sum_{\ell=1}^{\infty}\ell\sum_{j=1}^{\infty} S_j j\sum_{m=1}^{\infty}\frac{(-1)^{m+1}(\pi j)^{4m-2}\, a^{8m-4}}{(2m-1)\,(4m-1)\,!\,(4m-3)}\\
&\times\left.\left((\pi\ell)^{4m-2} S_\ell+\sum_{\mathrm{k}=1}^{\mathrm{m}}(-1)^{\ell+k+1}\frac{(4k-3)\,!}{2^{2k-2}}\,\frac{(\pi\ell)^{4m-4k}}{\sinh\,\pi\ell}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(\pi\ell)^{4n-2}\, a^{8n-4}}{2^{2n-1}(2n-1)\,(4n-1)\,!}\ldots.
\end{aligned}
\tag{4.2}
$$

Observe that the inverse procedure of the series under the condition $a \ll 1$,

$$1+2\,\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}+2\left(\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}\right)^2+2\left(\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}\right)^3+\ldots+,$$

employed into formula (4.2) yields the known MF formula

$$q_{\mathrm{MF}}=\frac{\lambda\left(1+\frac{\pi a^2}{4}\right)+1-\frac{\pi a^2}{4}}{\lambda\left(1-\frac{\pi a^2}{4}\right)+1+\frac{\pi a^2}{4}}. \tag{4.3}$$

Indeed, we have

$$\begin{aligned} q_{\mathrm{MF}}&=\frac{\lambda\left(1+\frac{\pi a^2}{4}\right)+1-\frac{\pi a^2}{4}}{\lambda\left(1-\frac{\pi a^2}{4}\right)+1+\frac{\pi a^2}{4}}=\frac{(\lambda+1)+(\lambda-1)\ \frac{\pi a^2}{4}}{(\lambda+1)-(\lambda-1)\ a^2}\,\frac{\pi a^2}{4}=\frac{1+\frac{\lambda-1}{\lambda+1}\ \frac{\pi a^2}{4}}{1-\frac{\lambda-1}{\lambda+1}\ \frac{\pi a^2}{4}}\\ &=\left(1+\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}\right)\left(1+\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}+\left(\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}\right)^2+\ \ldots\right)\\ &=1+2\,\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}+2\left(\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}\right)^2+2\left(\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}\right)^3+\ \ldots\ \ . \end{aligned}$$

Therefore, employment of the SAM allows to show analytically that the main term of the asymptotic series of the effective parameter of heat conductivity, in the case of cylindrical inclusions with circular cross sections having small sizes, yields the Maxwell formula (MF) and coincides both with the upper Hashin–Shtrikman bounds for $0 \le \lambda \le 1$ and with the lower Hashin–Shtrikman bounds for $1 \le \lambda < \infty$.

The first two nonzero correcting terms in the MF formula, accounting for the following approximation

$$1+\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}+\left(\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}\right)^2+\ldots=\frac{1}{1-\frac{\lambda-1}{\lambda+1}\ \frac{\pi a^2}{4}}=\frac{\lambda+1}{\lambda\left(1-\frac{\pi a^2}{4}\right)+1+\frac{\pi a^2}{4}}\quad \text{for } a\to 0$$

and based on the approximation (4.2), take the following form:

$$\begin{aligned} \Delta=&\left(\frac{\lambda-1}{\lambda+1}\right)^3\frac{\pi^2a^6}{4}\,\frac{(\lambda+1)^2}{\left(\lambda\left(1-\frac{\pi a^2}{4}\right)+1+\frac{\pi a^2}{4}\right)^2}\sum_{\ell=1}^{\infty}S_\ell\,\ell\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,(\pi\ell)^{4n-2}\,a^{8n-4}}{2^{2n-2}\,(2n-1)\ (4n-1)\,!}\\ &+\left(\frac{\lambda-1}{\lambda+1}\right)^4\frac{\pi^3a^8}{16}\,\frac{(\lambda+1)^2}{\left(\lambda\left(1-\frac{\pi a^2}{4}\right)+1+\frac{\pi a^2}{4}\right)^2}\sum_{\ell=1}^{\infty}\ell\sum_{j=1}^{\infty}S_j\,j\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\,(\pi j)^{4m-2}\,a^{8m-4}}{(2m-1)\ (4m-1)\,!\,(4m-3)}\\ &\times\left((\pi\ell)^{4m-2}\,S_\ell+\sum_{k=1}^{\mathrm{m}}(-1)^{\ell+k+1}\,\frac{(4k-3)\,!}{2^{2k-2}}\,\frac{(\pi\ell)^{4m-4k}}{\sinh\,\pi\ell}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,(\pi\ell)^{4n-2}\,a^{8n-4}}{2^{2n-1}\,(2n-1)\ (4n-1)\,!}. \end{aligned} \tag{4.4}$$

Consequently, the final form of the effective heat transfer coefficient treated as the generalized (N-iterated) solution of the SAM problem is recast in the following form

$$\begin{aligned} q_{\mathrm{SAM}\,(N)}=&\frac{\lambda\left(1+\frac{\pi a^2}{4}\right)+1-\frac{\pi a^2}{4}}{\lambda\left(1-\frac{\pi a^2}{4}\right)+1+\frac{\pi a^2}{4}}+\left(\frac{\lambda-1}{\lambda+1}\right)^3\,\frac{\pi^2a^6}{4}\,\frac{(\lambda+1)^2}{\left(\lambda\left(1-\frac{\pi a^2}{4}\right)+1+\frac{\pi a^2}{4}\right)^2}\\ &\times\left(\Delta_1+\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}\,\Delta_2\right)\\ =&\,q_{\mathrm{MF}}+\frac{\lambda-1}{\lambda+1}\,\frac{\pi^2a^6}{4}\,\frac{(\lambda-1)^2}{\left(\lambda\left(1-\frac{\pi a^2}{4}\right)+1+\frac{\pi a^2}{4}\right)^2}\left(\Delta_1+\frac{\lambda-1}{\lambda+1}\,\frac{\pi a^2}{4}\,\Delta_2\right), \end{aligned} \tag{4.5}$$

where

$$\Delta_1 = \sum_{\ell=1}^{\infty} S_\ell \ell \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (\pi\ell)^{4n-2} a^{8n-4}}{2^{2n-2} (2n-1)\ (4n-1)\,!};$$

$$\Delta_2 = \sum_{\ell=1}^{\infty} \ell \sum_{j=1}^{\infty} S_j j \sum_{m=1}^{\infty} \frac{(-1)^{m+1} (\pi j)^{4m-2} a^{8m-4}}{(2m-1)\ (4m-1)\,!\,(4m-3)\,!}$$

$$\times \left((\pi\ell)^{4m-2} S_\ell + \sum_{k=1}^{\mathrm{m}} (-1)^{\ell+k+1} \frac{(4k-3)\,!}{2^{2k-2}} \frac{(\pi\ell)^{4m-4k}}{\sinh\,\pi\ell} \right) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (\pi\ell)^{4n-2} a^{8n-4}}{2^{2n-1} (2n-1)\ (4n-1)\,!}. \quad (4.6)$$

It should be noticed that the series occurring in (4.4) of the form

$$T^{(i)} = \sum_{n=1}^{\infty} S_n n^i$$

are rapidly convergent because the following estimation holds:

$$\frac{T_{n+1}^{(i)}}{T_n^{(i)}} = \frac{S_{n+1}^{(i)}\ (n+1)^i}{S_n^{(i)} n^i} \sim e^{-\pi}.$$

Thus, summing the series with regard to 5 and 100 terms of the series terms $T^{(3)}$, $T^{(5)}$, $T^{(7)}$, we get

$$\begin{aligned} &T^{(3)}\ (5) \approx 0.0190558992; \quad T^{(3)}\ (100) \approx 0.0190558917;\\ &T^{(5)}\ (5) \approx 0.0177068353; \quad T^{(5)}\ (100) \approx 0.0177065687;\\ &T^{(7)}\ (5) \approx 0.0138798059; \quad T^{(7)}\ (100) \approx 0.0138704300. \end{aligned}$$

In the case of small sizes of the inclusions, the obtained asymptotic formula of the effective parameter $q_{\mathrm{SAM}\,(N)}$ (4.5) satisfies the Keller's theorem [8]

$$q_{\mathrm{SAM}\,(N)}\ (\lambda) = q^{-1}_{\mathrm{SAM}\,(N)}\ \left(\lambda^{-1}\right),$$

if one considers accuracy of the series of order a^{14}.

Indeed, let us recast formulas (4.6) into the following form:

$$\Delta_1 = \sum_{\ell=1}^{\infty} S_\ell \ell \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ 2^{6n-2}\ \ell^{4n-2}}{(2n-1)\ (4n-1)\,!} \left(\frac{\pi a^2}{4}\right)^{4n-2} = \sum_{n=1}^{\infty} \delta_1^{(n)} \left(\frac{\pi a^2}{4}\right)^{4n-2};$$

$$\Delta_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n+2}\ \pi^{4(m+n-1)} a^{8(m+n-1)}}{(2m-1)\ (4m-1)\,!\,(4m-3)\,!\,2^{2n-1}\ (2n-1)\ (4n-1)\,!} \sum_{\ell=1}^{\infty} \ell^{4n-1} \sum_{j=1}^{\infty} S_{\mathrm{j}} j^{4\mathrm{m}-1}$$

$$\times \left((\pi\ell)^{4m-2} S_\ell + \sum_{k=1}^{\mathrm{m}} (-1)^{\ell+k+1} \frac{(4k-3)\,!}{2^{2k-2}} \frac{(\pi\ell)^{4m-4k}}{\sinh\,\pi\ell} \right)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \delta_2^{(mn)} \left(\frac{\pi a^2}{4}\right)^{4(m+n-1)}, \quad (4.7)$$

where

$$\delta_1^{(n)} = \frac{(-1)^{n+1}\ 2^{6n-2}}{(2n-1)\ (4n-1)\,!} \sum_{\ell=1}^{\infty} S_\ell \ell^{4n-1};$$

$$\delta_2^{(mn)} = \frac{(-1)^{m+n+2}\ 2^{8m+6n-7}}{(2m-1)\ (4m-1)\,!\,(4m-3)\,!\ (2n-1)\ (4n-1)\,!} \sum_{\ell=1}^{\infty} \ell^{4n-1} \sum_{j=1}^{\infty} S_{\mathrm{j}} j^{4m-1}$$

$$\times\left((\pi\ell)^{4m-2}\,S_\ell+\sum_{k=1}^{m}(-1)^{\ell+k+1}\,\frac{(4k-3)\,!}{2^{2k-2}}\,\frac{(\pi\ell)^{4m-4k}}{\sinh\,\pi\ell}\right). \tag{4.8}$$

We have

$$\begin{aligned}
q_{\mathrm{SAM}(N)}(\lambda) &= \frac{1+\frac{\lambda-1}{\lambda+1}\frac{\pi a^2}{4}}{1-\frac{\lambda-1}{\lambda+1}\frac{\pi a^2}{4}}+\frac{\left(\frac{\lambda-1}{\lambda+1}\right)^3}{\left(1-\frac{\lambda-1}{\lambda+1}\frac{\pi a^2}{4}\right)^2}\frac{\pi^2a^6}{4}\\
&\times\left[\delta_1^{(1)}\frac{\pi^2a^4}{4^2}+\delta_1^{(2)}\frac{\pi^6a^{12}}{4^6}+\ldots+\frac{\lambda-1}{\lambda+1}\frac{\pi a^2}{4}\left(\delta_2^{(11)}\frac{\pi^4a^8}{4^4}+\left(\delta_2^{(12)}+\delta_2^{(21)}\right)\frac{\pi^8a^{16}}{4^8}+\ldots\right)\right]\sim\\
&\sim 1+2\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}a^2+2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}\right)^2a^4+2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}\right)^3a^6+2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi a^2}{4}\right)^4a^8\\
&+\left(2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}\right)^5+\left(\frac{\lambda-1}{\lambda+1}\right)^2\frac{\pi^4}{4^3}\delta_1^{(1)}\right)a^{10}+\left(2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}\right)^6+2\left(\frac{\lambda-1}{\lambda+1}\right)^4\frac{\pi^5}{4^4}\delta_1^{(1)}\right)a^{12}\\
&+\left(2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}\right)^7+3\left(\frac{\lambda-1}{\lambda+1}\right)^5\frac{\pi^6}{4^5}\delta_1^{(1)}\right)a^{14}\\
&+\left(2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}\right)^8+4\left(\frac{\lambda-1}{\lambda+1}\right)^6\frac{\pi^7}{4^6}\delta_1^{(1)}+\left(\frac{\lambda-1}{\lambda+1}\right)^4\frac{\pi^7}{4^6}\delta_2^{(11)}\right)a^{16}+o\left(a^{16}\right)\ \text{for } a\to 0;
\end{aligned}\tag{4.9}$$

$$\begin{aligned}
q^{-1}_{\mathrm{SAM}(N)}\left(\lambda^{-1}\right) &= \left\{\frac{1-\frac{\lambda-1}{\lambda+1}\frac{\pi a^2}{4}}{1+\frac{\lambda-1}{\lambda+1}\frac{\pi a^2}{4}}-\frac{\left(\frac{\lambda-1}{\lambda+1}\right)^3}{\left(1+\frac{\lambda-1}{\lambda+1}\frac{\pi a^2}{4}\right)^2}\frac{\pi^2a^6}{4}\right.\\
&\left.\times\left[\delta_1^{(1)}\frac{\pi^2a^4}{4^2}+\delta_1^{(2)}\frac{\pi^6a^{12}}{4^6}+\ldots-\frac{\lambda-1}{\lambda+1}\frac{\pi a^2}{4}\left(\delta_2^{(11)}\frac{\pi^4a^8}{4^4}+\left(\delta_2^{(12)}+\delta_2^{(21)}\right)\frac{\pi^8a^{16}}{4^8}+\ldots\right)\right]\right\}^{-1}\\
&\sim 1+2\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}a^2+2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}\right)^2a^4+2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}\right)^3a^6+2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi a^2}{4}\right)^4a^8\\
&+\left(2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}\right)^5+\left(\frac{\lambda-1}{\lambda+1}\right)^2\frac{\pi^4}{4^3}\delta_1^{(1)}\right)a^{10}+\left(2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}\right)^6+2\left(\frac{\lambda-1}{\lambda+1}\right)^4\frac{\pi^5}{4^4}\delta_1^{(1)}\right)a^{12}\\
&+\left(2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}\right)^7+3\left(\frac{\lambda-1}{\lambda+1}\right)^5\frac{\pi^6}{4^5}\delta_1^{(1)}\right)a^{14}\\
&+\left(2\left(\frac{\lambda-1}{\lambda+1}\frac{\pi}{4}\right)^8+4\left(\frac{\lambda-1}{\lambda+1}\right)^6\frac{\pi^7}{4^6}\delta_1^{(1)}-\left(\frac{\lambda-1}{\lambda+1}\right)^4\frac{\pi^7}{4^6}\delta_2^{(11)}\right)a^{16}+o\left(a^{16}\right)\ \text{for } a\to 0.
\end{aligned}\tag{4.10}$$

Comparison of relations (4.9) and (4.10) implies the following conclusion. In the solutions (4.5), (4.6), found based on the SAM, the sum of the main part of the asymptotic representation q_{MF} and the first correcting term Δ_1

$$q^{(1)}_{\mathrm{SAM}(N)}=\frac{\lambda\left(1+\frac{\pi a^2}{4}\right)+1-\frac{\pi a^2}{4}}{\lambda\left(1-\frac{\pi a^2}{4}\right)+1+\frac{\pi a^2}{4}}+\left(\frac{\lambda-1}{\lambda+1}\right)^3\frac{\pi^2a^6}{4}\frac{(\lambda+1)^2}{\left(\lambda\left(1-\frac{\pi a^2}{4}\right)+1+\frac{\pi a^2}{4}\right)^2}\Delta_1$$

satisfies the Keller theorem up to the terms of order a^{2n} inclusively for arbitrary vales of n. Though the error is introduced already by the second correcting term Δ_2, its order is of a^{16} for small a.

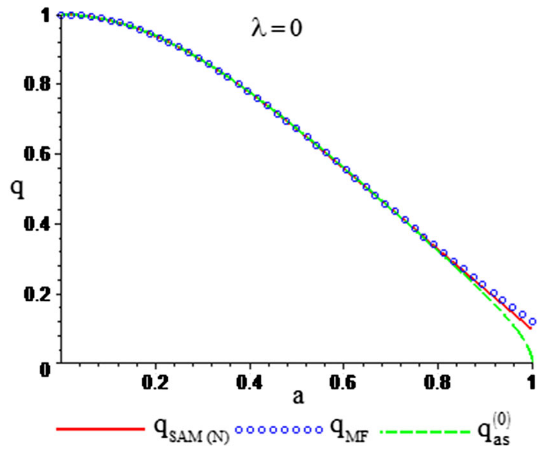

Fig. 1 Effective heat transfer coefficient for non-conductive inclusions $\lambda = 0$: $q_{\mathrm{SAM}(N)}$ – generalized SAM; q_{MF} – MF; $q_{\mathrm{as}}^{(0)}$ – asymptotic solutions

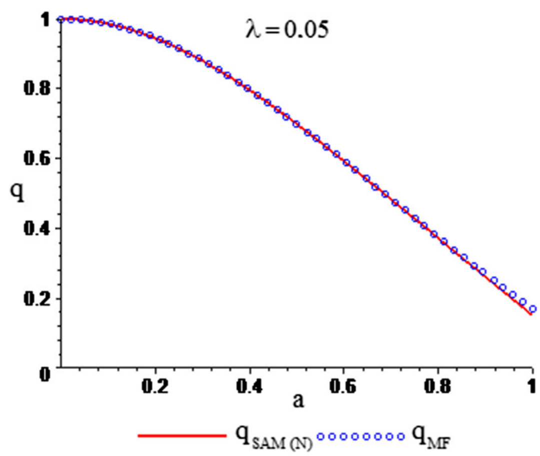

Fig. 2 Effective heat transfer coefficient of inclusions with small conductivity $\lambda = 0.05$: $q_{\mathrm{SAM}(N)}$ – generalized SAM; q_{MF} – MF;

5 Analysis of the obtained corrections to the MF

In formula (4.5), the MF relation (4.3) stands for the main part of the asymptotic form of the effective parameter, and Δ_1, Δ_2 play the role of correcting terms of order a^{10} and higher, and they are defined via series regarding m and n. For instance, for $n = 1$, the order of correcting term is a^{10}.

Let us consider the case study, when we take $n = 2$ in formulas (4.6) and include the terms of order a^{18}, and we estimate through formulas (4.5), (4.6) the effective heat transfer parameter for various values of the sizes of inclusions and their conductivity (see results reported in Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9).

6 Numerical results

The so far reported graphical results allow to formulate the following observations:

1. MF describes well the averaged parameter of a structure for small and average sizes of inclusions (up to $a \approx 0.7$) for arbitrary inclusions accounting for the limiting cases ($\lambda = 0$ and $\lambda \to \infty$). In the latter case, computations carried out with MF and generalized SAM practically coincide (Table 1).
2. In the case when the conductivity of the matrix and the inclusion is of the same order (from $\lambda \approx 0.5$ up to $\lambda \approx 2$) the computational results regarding the effective parameter estimated due to MF and SAM are very close to each other (Table 2) in the whole interval of the inclusion size ($0 \le a \le 1$).

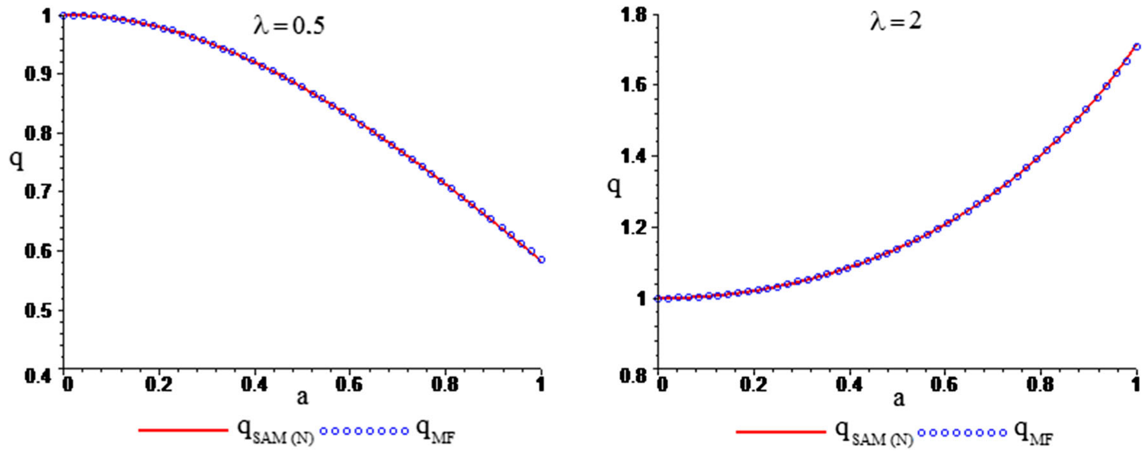

Fig. 3 The effective heat transfer coefficient of the heat transfer for the case of conductivity of inclusions of the order of the matrix conductivity: $q_{\mathrm{SAM}(N)}$ – generalized SAM; q_{MF} – MF for $\lambda = 0.5$ and for $\lambda = 2$

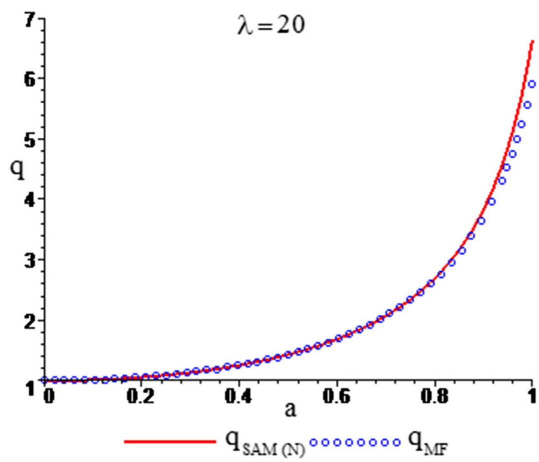

Fig. 4 Effective heat transfer coefficient for inclusions with high conductivity $\lambda = 20$: $q_{\mathrm{SAM}(N)}$ – generalized SAM; q_{MF} – MF

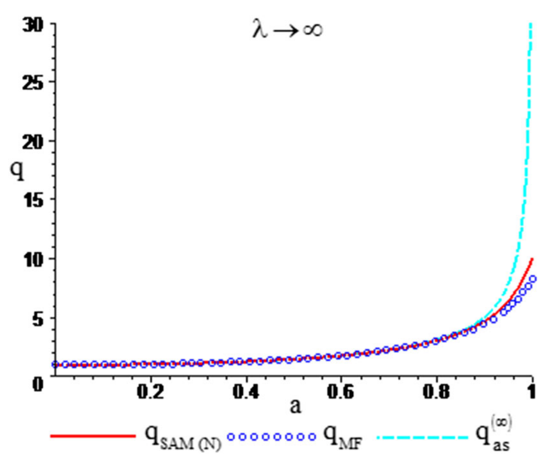

Fig. 5 Effective heat transfer coefficient for absolute conductive inclusions $\lambda \to \infty$: $q_{\mathrm{SAM}(N)}$ – generalized SAM; q_{MF} – MF; $q_{\mathrm{as}}^{(\infty)}$ – asymptotic solution

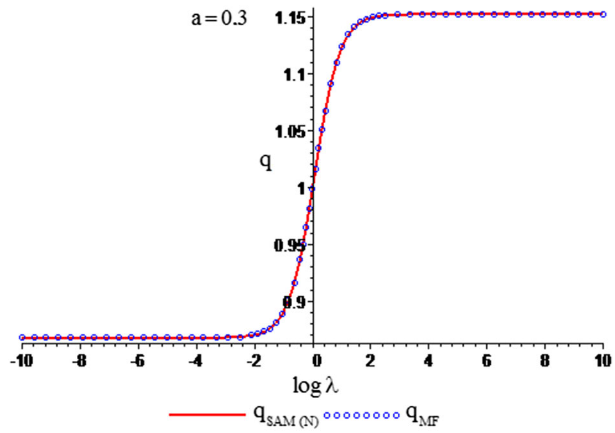

Fig. 6 Effective heat transfer coefficient for inclusions of small size $a = 0.3$: $q_{\mathrm{SAM}(N)}$ – SAM; q_{MF} – MF

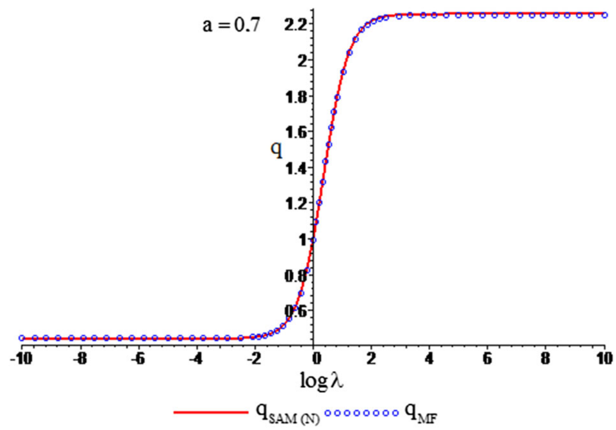

Fig. 7 Effective heat transfer coefficient for inclusions of average size $a = 0.7$: $q_{\mathrm{SAM}(N)}$ – generalized SAM; q_{MF} – MF

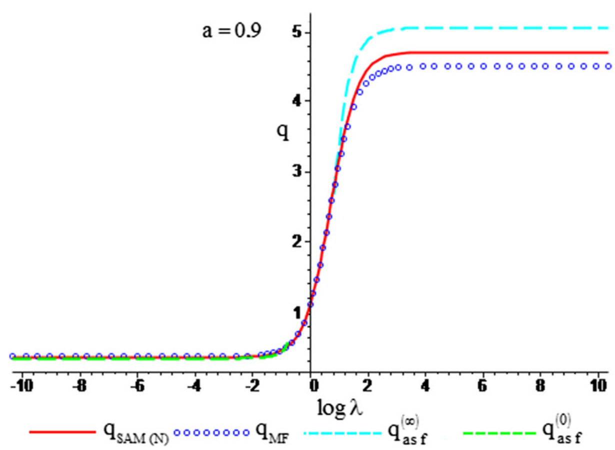

Fig. 8 Effective heat transfer coefficient for large size inclusions $a = 0.9$: $q_{\mathrm{SAM}(N)}$ – generalized SAM; q_{MF} – MF; $q_{\mathrm{asf}}^{(\infty)}$ – asymptotic formula (49) from [7]; $q_{\mathrm{asf}}^{(0)}$ – asymptotic formula (49) from [7], transformed on the basis of Keller's theorem [8]

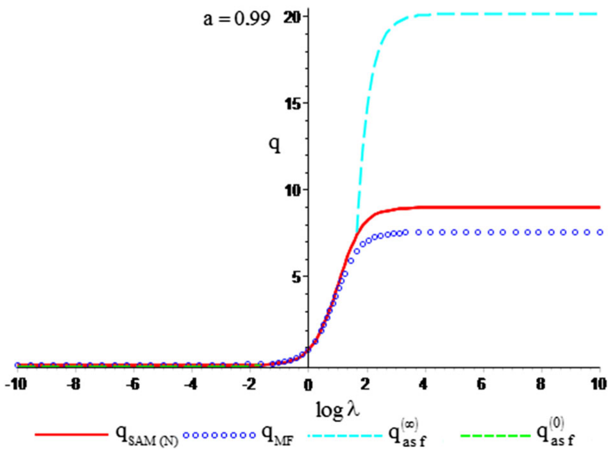

Fig. 9 Effective heat transfer coefficient for the case of nearly limiting value for the inclusions size $a = 0.99$: $q_{\mathrm{SAM}(N)}$ – generalized SAM; q_{MF} – MF; $q_{\mathrm{asf}}^{(\infty)}$ – asymptotic formula (49) from [7]; $q_{\mathrm{asf}}^{(0)}$ – asymptotic formula (49) from [7], transformed on the basis of Keller's theorem [8]

Table 1 Conductivity of inclusions: MF q_{MF} versus SAM $q_{SAM(N)}$ for different a ($\lambda = 0$ and $\lambda \to \infty$)

Size of inclusions a	MF q_{MF}	SAM $q_{\mathrm{SAM}(N)}$	Error $\frac{q_{\mathrm{MF}}-q_{\mathrm{SAM}(N)}}{q_{\mathrm{MF}}} \cdot 100\%$
Conductivity of inclusions $\lambda = 0$			
0.1	0.9844	0.9844	0.7735×10^{-9}
0.2	0.9391	0.9391	0.7928×10^{-6}
0.3	0.8680	0.8680	0.4589×10^{-4}
0.4	0.7767	0.7767	0.8238×10^{-3}
0.5	0.6718	0.6717	0.7847×10^{-2}
0.6	0.5592	0.5589	0.5065×10^{-1}
0.7	0.4442	0.4431	0.2541
Conductivity of inclusions $\lambda = 0.05$			
0.1	0.9859	0.9859	0.5729×10^{-9}
0.2	0.9447	0.9447	0.5870×10^{-6}
0.3	0.8798	0.8798	0.3396×10^{-4}
0.4	0.7958	0.7958	0.6083×10^{-3}
0.5	0.6983	0.6983	0.5770×10^{-2}
0.6	0.5926	0.5924	0.3694×10^{-1}
0.7	0.4835	0.4826	0.1826
Conductivity of inclusions $\lambda = 20$			
0.1	1.0143	1.0143	-0.5729×10^{-9}
0.2	1.0585	1.0585	-0.5870×10^{-6}
0.3	1.1366	1.1366	-0.3397×10^{-4}
0.4	1.2566	1.2566	-0.6087×10^{-3}
0.5	1.4321	1.4321	-0.5782×10^{-2}
0.6	1.6875	1.6881	-0.3717×10^{-1}
0.7	2.0684	2.0722	−0.1855
Conductivity of inclusions $\lambda \to \infty$			
0.1	1.0158	1.0158	-0.7735×10^{-9}
0.2	1.0649	1.0649	0.7928×10^{-6}
0.3	1.1521	1.1521	-0.4590×10^{-4}
0.4	1.2874	1.2875	-0.8243×10^{-3}
0.5	1.4886	1.4888	-0.7865×10^{-2}
0.6	1.7884	1.7893	-0.5100×10^{-1}
0.7	2.2512	2.2570	−0.2586

Table 2 Conductivity of inclusions: MF q_{MF} versus SAM $q_{\mathrm{SAM}(N)}$ for different a ($\lambda \in (0.5; 2)$

Size of inclusions a	MF q_{MF}	SAM $q_{\mathrm{SAM}(N)}$	Error $\frac{q_{\mathrm{MF}}-q_{\mathrm{SAM}(N)}}{q_{\mathrm{MF}}} \cdot 100\%$
Conductivity of inclusions $\lambda = 0.5$			
0.1	0.9948	0.9948	0.2865×10^{-10}
0.2	0.9793	0.9793	0.2934×10^{-7}
0.3	0.9540	0.9540	0.1692×10^{-5}
0.4	0.9196	0.9196	0.3009×10^{-4}
0.5	0.8771	0.8771	0.2808×10^{-3}
0.6	0.8277	0.8277	0.1745×10^{-2}
0.7	0.7726	0.7725	0.8200×10^{-2}
0.8	0.7130	0.7128	0.3140×10^{-1}
0.9	0.6501	0.6494	0.1029
1.0	0.5850	0.5833	0.2978
Conductivity of inclusions $\lambda = 2$			
0.1	1.0052	1.0052	-0.2865×10^{-10}
0.2	1.0212	1.0212	-0.2934×10^{-7}
0.3	1.0483	1.0483	-0.1692×10^{-5}
0.4	1.0874	1.0874	-0.3009×10^{-4}
0.5	1.1401	1.1401	-0.2810×10^{-3}
0.6	1.2081	1.2081	-0.1749×10^{-2}
0.7	1.2943	1.2944	-0.8247×10^{-2}
0.8	1.4026	1.4030	-0.3181×10^{-1}
0.9	1.5383	1.5399	−0.1056
1.0	1.7093	1.7146	−0.3129

Table 3 Conductivity of inclusions: MF q_{MF} versus SAM $q_{\mathrm{SAM}(N)}$ for different a ($\lambda > 10$ or $\lambda < 0.1$)

Size of inclusions a	MFq_{MF}	SAM $q_{\mathrm{SAM}(N)}$	Error $\frac{q_{\mathrm{MF}}-q_{\mathrm{SAM}(N)}}{q_{\mathrm{MF}}} \cdot 100\%$
Conductivity of inclusions $\lambda = 0.05$			
0.80	0.3748	0.3719	0.7610
0.85	0.3215	0.3167	1.4899
0.90	0.2694	0.2616	2.8720
0.95	0.2185	0.2065	5.5130
1.00	0.1692	0.1510	10.7180
Conductivity of inclusions $\lambda = 20$			
0.80	2.6683	2.6893	−0.7881
0.85	3.1102	3.1590	−1.5670
0.90	3.7124	3.8268	−3.0837
0.95	4.5759	4.8543	−6.0841
1.00	5.9108	6.6355	−12.2612

Table 4 Conductivity of inclusions: MF q_{MF} versus SAM $q_{\mathrm{SAM}(N)}$ for ($a \to 1,\ \lambda \to \infty$ or $\lambda \to 0$)

Size of inclusions a	MF q_{MF}	SAM $q_{\mathrm{SAM}(N)}$	Asymptotic solution $q_{\mathrm{as}}^{(0)}$	Error $\frac{q_{\mathrm{as}}^{(0)}-q_{\mathrm{MF}}}{q_{\mathrm{as}}^{(0)}} \cdot 100\%$	Error $\frac{q_{\mathrm{as}}^{(0)}-q_{\mathrm{SAM}(N)}}{q_{\mathrm{as}}^{(0)}} \cdot 100\%$
Conductivity of inclusions $\lambda = 0$					
0.80	0.3310	0.3274	0.3232	−2.4176	−1.3028
0.85	0.2760	0.2699	0.2616	−5.4749	−3.1765
0.90	0.2224	0.2127	0.1974	−12.6438	−7, 7561
0.95	0.1704	0.1555	0.1263	−34.9469	−23.1268
0.99	0.1301	0.1095	0.0497	−161.8549	−120.3139
Conductivity of inclusions $\lambda \to \infty$					
0.80	3.0214	3.0555	3.0944	2.3605	1.2558
0.85	3.6237	3.7072	3.8221	5.1908	3.0064
0.90	4.4971	4.7082	5.0657	11.2246	7.05759
0.95	5.8686	6.4419	7.9196	25.8968	18.6587
0.99	7.6869	9.0895	20.1286	61.8019	54.8427

3. In the case of large inclusion size ($a > 0.8$) as well as either large ($\lambda > 10$) or small ($\lambda < 0.1$) conductivity, the correcting term yielded by SAM (Table 3) plays an essential role.

4. In the case of large size of inclusions close to the limiting value ($a \to 1$), and for large ($\lambda \to \infty$) and small ($\lambda \to 0$) limiting values of the conductivity, neither MF nor SAM offer a reliable picture of the behavior of the effective parameter. This observation is confirmed by the divergence of the results shown in Table 4 versus the values of the known asymptotic solution (1.2) and its counterpart formula (1.3).

7 Concluding remarks

In this work, we have proposed new formulas for estimation of the effective coefficient of thermal conductivity based on the method of asymptotic homogenization. Our research allows for the following conclusions:

1. The MF asymptotic analysis of a two-phase composite model for the case of a composite structure with periodically distributed cylindrical inclusions of circular cross sections was carried out.
2. The solution has been constructed with the help of averaging theory SAM and PA.
3. Employment of the generalized (N-iterations) SAM procedure allowed to conclude that for small sizes of inclusions, the main part of the asymptotic series of the effective heat transfer parameter coincides with the MF. It has been illustrated that development of the correcting term into series with regard to the MF begins with terms of order a^{10}. We have proposed relations allowing for improvement of the results up to terms of order a^{18}.
4. Analysis of the solution based on the generalized SAM procedure yielded the following important issues:
   (a) In the case of small and average sizes of inclusions and their arbitrary conductivity, the MF formula represents well the averaged parameter of the structure; the computation results practically coincide with the solutions obtained by the SAM.
   (b) In the case of composites with conductivity of inclusions close to the matrix conductivity, the computational results of the effective parameter estimated through MF and SAM are close to each other in the whole interval of the inclusion sizes.
   (c) It is recommended to take into account the correcting term given by the N-iteration SAM solution while investigating the effective conductivity of structures with large size of inclusions or their small conductivity.
   (d) In the case of the limiting large sizes of inclusions and conductivity, both MF and SAM do not offer reliable results. Namely, they do not describe neither qualitatively nor quantitatively the behavior of the effective parameter and physically result in the emergence of an infinite cluster.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Maxwell, J.C.: Treatise on Electricity and Magnetism. Clarendon Press, Oxford (1873)
2. Landauer, R.: The electrical resistance of binary metallic mixture. J. Appl. Phys. **23**(7), 779–784 (1952)
3. Lichtenecker, K.: Die Dielektrizitätskonstante natürlicher und künstlicher Mischkörper. Phys. Zeits. **27**(4), 115–158 (1926)
4. Shvidler, M.I.: Statistical Hydrodynamics of Porous Media. Nedra, Moscow (1985). (in Russian)
5. Milton, G.W.: The Theory of Composites. Cambridge University Press, Cambridge (2002)

6. Torquato, S.: Random Heterogeneous Materials: Microstructure and Macroscopic Properties. Springer, New York (2002)
7. McPhedran, R.C., Poladian, L., Milton, G.W.: Asymptotic studies of closely spaced, highly conducting cylinders. Proc. R. Soc. A **415**, 185–196 (1988)
8. Keller, J.B.: A theorem on the conductivity of a composite medium. J. Math. Phys. **5**(4), 548–549 (1964)
9. Gluzman, S., Mityushev, V., Nawalaniec, W.: Computational Analysis of Structured Media. Elsevier, Amsterdam (2017)
10. Drygaś, P., Gluzman, S., Mityushev, V., Nawalaniec, W.: Applied Analysis of Composite Media. Analytical and Computational Results for Materials Scientists and Engineers. Elsevier, Amsterdam (2020)
11. Christensen, R.M.: Mechanics of Composite Materials. Dover Publications, Mineola (2005)
12. Andrianov, I.V., Kalamkarov, A.L., Starushenko, G.A.: Analytical expressions for effective thermal conductivity of composite materials with inclusions of square cross-section. Compos. Part B Eng. **50**, 44–53 (2013)
13. Andrianov, I.V., Kalamkarov, A.L., Starushenko, G.A.: Three-phase model for a fiber-reinforced composite material. Compos. Struct. **95**, 95–104 (2013)
14. Kalamkarov, A.L., Andrianov, I.I., Starushenko, G.A.: Refinement of the Maxwell formula for fibre-reinforced composites. J. Multiscale Model. **10**(1), 1950001-1–1950001-33 (2019)
15. Berdichevsky, V.L.: Variational Principles of Mechanics of Continuous Media. Nauka, Moscow (1983). (in Russian)
16. Mityushev, V., Rylko, N.: Maxwells approach to effective conductivity and its limitations. Q. J. Mech. Appl. Math. **66**(2), 241–251 (2013)
17. Berlyand, L., Mityushev, V.: Generalized Clausius-Mossotti formula for random composite with circular fibers. J. Stat. Phys. **102**(1–2), 115–145 (2001)
18. Levin, V., Kanaun, S., Markov, M.: Generalized Maxwell's scheme for homogenization of poroelastic composites. Int. J. Eng. Sci. **61**, 75–86 (2012)
19. Mityushev, V.: Transport properties of double-periodic arrays of circular cylinders. ZAMM **77**(2), 115–120 (1997)
20. Mityushev, V.: Cluster method in composites and its convergence. Appl. Math. Lett. **77**, 44–48 (2018)
21. Mityushev, V., Drygas, P.: Effective properties of fibrous composites and cluster convergence. Multiscale Model. Simul. **17**(2), 696–715 (2019)
22. Mityushev, V.: Random 2D composites and the generalized method of Schwarz. Adv. Math. Phys. **2015**, 15 (2015)
23. Mityushev, V., Nawalaniec, W.: Effective conductivity of a random suspension of highly conducting spherical particles. Appl. Math. Model. **72**, 230–246 (2019)
24. Schwarz, H.A.: Über einige Abbildungsaufgaben. Ges. Math. Abh. **II**(11), 65–83 (1869)
25. Courant, R.: Geometrische Funktionentheorie. Springer, Berlin (1963)
26. Kantorovich, L.V., Krylov, V.I.: Approximate Methods of Higher Analysis. Noordhoff, Groningen (1958)
27. Mikhlin, S.G.: Integral Equations. Pergamon Press, Oxford (1964)
28. Abramowitz, M., Stegun, I.A. (eds.): Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables. Dover Publications, New York (1965)