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GENERATING CHAOS IN 3D SYSTEMS OF QUADRATIC DIFFERENTIAL EQUATIONS WITH 1D EXPONENTIAL MAPS

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New existence conditions of homoclinic orbits for some systems of ordinary quadratic differential equations with singular linear part are found. A realization of these conditions guarantees the existence of chaotic attractors at 3D autonomous quadratic systems. In addition, a chaotic behavior of solutions of these systems is determined by the 1D discrete map $x_{n+1} = rx_n \cdot [\exp(px_n - x_n^2)]/(1 + \gamma x_n)$ at some values of parameters $r > 0$, $p \in \mathbb{R}$, and $\gamma \in (-d, \infty)$, where $d > 0$; $n = 0, 1, 2, \ldots$. Examples of chaotic attractors are given.

Keywords: 1D discrete map; ordinary autonomous differential equations system; limit cycle; homoclinic orbit; chaotic attractor.

1. Introduction

Chaos as a very interesting complex nonlinear phenomenon has been intensively studied in the last four decades within the science, mathematics and engineering communities. Recently, chaos has been found to be very useful and has great potential in many technological disciplines, such as information and computer sciences, power systems protection, biomedical systems analysis, flow dynamics and liquid mixing, encryption and communications, and so on. It is not surprising, therefore, that academic researches on chaotic dynamics has evolved from traditional trends of analyzing and understanding chaos to new directions of controlling and utilizing it.

An open question of chaos theory is: What discrete processes are sources of a chaotic behavior for continuous dynamic systems? Some answers for this question will be given below in the present work.

Basic ideas and methods, which will be developed in our paper, rise from [Belozyorov, 2007].

We denote by $\mathbb{R}^n$ a real space of dimension n. Let $\mathbf{x}^T = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be an unknown vector, where coordinates $x_i = x_i(t)$ are functions of time t. Let also $A = (a_{ij}), B_1, \ldots, B_n \in \mathbb{R}^{n\times n}$ be real matrices and let the matrices $B_1, \ldots, B_n$ be symmetrical.

Consider the system of ordinary quadratic differential equations

$$\begin{cases} \dot{x}_1(t) = \sum_{j=1}^{n} a_{1j}x_j(t) + \mathbf{x}^T(t)B_1\mathbf{x}(t) \equiv f_1(\mathbf{x}(t)), \\ \quad\vdots \\ \dot{x}_n(t) = \sum_{j=1}^{n} a_{nj}x_j(t) + \mathbf{x}^T(t)B_n\mathbf{x}(t) \equiv f_n(\mathbf{x}(t)) \end{cases} \tag{1}$$

of order n with the vector of initial values $\mathbf{x}^T(0) = (x_{10}, \ldots, x_{n0})$.

It is well known that the linearization method is a basic method for the research in nonlinear systems. In [Belozyorov, 2007] another method for the research of system (1) was offered. The essence of this method consists of the following. First the system

$$\begin{cases} \dot{x}_1(t) = \mathbf{x}^T(t)B_1\mathbf{x}(t), \\ \quad\vdots \\ \dot{x}_n(t) = \mathbf{x}^T(t)B_n\mathbf{x}(t) \end{cases} \tag{2}$$

is investigated. After that, an influence of elements $\sum a_{ij}x_j$ on the solutions of system (2) was examined. Namely, with this approach, new results on the boundedness of the solutions of system (1) were derived. In particular, for new conditions of the existence of homoclinic orbits in system (1), the mentioned approach has been used in [Belozyorov, 2011a].

Today the most known results devoted to chaotic dynamics in autonomous 3D systems of differential equations are based on the supposition of the existence in these systems of either homoclinic or heteroclinic orbits, and the use of Shilnikov Theorem (see, for example, [Li *et al.*, 2004; Qi *et al.*, 2008; Shang & Han, 2005; Wang, 2009; Zhou *et al.*, 2004; Zheng & Chen, 2006; Zhou & Chen, 2006], and many references cited therein).

We especially remark on [Zhou & Chen, 2006] in which by the undetermined coefficients method, an existence of heteroclinic orbit (it means the existence of chaotic dynamics) was rigorously proved for the famous Lorenz system

$$\begin{cases} \dot{x}(t) = a(y(t) - x(t)), \\ \dot{y}(t) = cx(t) - x(t)z(t) - y(t), \\ \dot{z}(t) = x(t)y(t) - bz(t). \end{cases} \tag{3}$$

(Here $a = 10, b = 8/3, c = 28$.)

Note that in all indicated works it was assumed that the Jacobian matrix in any equilibrium point was not singular. The first result, in which the last condition was ignored (the Jacobian matrix is assumed to be singular), was represented in [Chen *et al.*, 2009]. However, the conditions guaranteeing the existence of the homoclinic orbit connected at some equilibrium point in [Chen *et al.*, 2009] were not indicated. Its presence was simple postulated.

In the present paper, for the general quadratic systems with singular linear part, constructive conditions for the existence of homoclinic orbits are seen. Examples of new chaotic attractors are shown. In addition, the connection of these conditions is determined with the existence of an 1D discrete map, generating a chaotic dynamics in the considered autonomous systems of differential equations.

It should be mentioned that for general quadratic systems 1D implicit discrete maps were built in [Belozyorov, 2011b, 2012]. In the present paper, the 1D discrete map generating chaos in the quadratic 3D system (1) is built in an explicit form.

Let us introduce some notations and definitions. Let $Q \subset \mathbb{R}^n$ be a compact (bounded and closed) set containing the origin. Symbol $\mathbf{x}(t, \mathbf{x}_0)$ denotes the solution (the trajectory) of system (1) satisfying the initial condition $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$. Further, we denote the distance between any vector $\mathbf{x}_k$ and Q by $d(\mathbf{x}_k, Q) = \inf_{\mathbf{x}\in Q} \|\mathbf{x}_k - \mathbf{x}\|$.

Definition 1 [Belozyorov, 2011a]. If there exists a compact set $Q \subset \mathbb{R}^n$ such that

$$\forall\, \mathbf{x}_0 \in \mathbb{R}^n, \quad \lim_{t\to\infty} d(\mathbf{x}(t, \mathbf{x}_0), Q) = 0,$$

then we call Q a globally attractive set of system (1). If

$$\forall\, \mathbf{x}_0 \in P \subset \mathbb{R}^n \;\Rightarrow\; \mathbf{x}(t, \mathbf{x}_0) \subseteq P, \quad \forall\, t \geq 0,$$

then P is called a positive invariant set of system (1).

Definition 2 [Zhou & Chen, 2006]. A bounded trajectory $\mathbf{x}(t, \mathbf{x}_0)$ of system (1) is called a homoclinic orbit if the trajectory converges to the same equilibrium point as $t \to \pm\infty$.

Let $\mathbf{x}_e \in \mathbb{R}^n$ be an equilibrium point of system (1). Denote by

$$D(\mathbf{x}_e) = \left(\frac{\partial f_i(\mathbf{x})}{\partial x_j}\right)(\mathbf{x}_e) \in \mathbb{R}^{n\times n}$$

the Jacobian matrix of the function $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))^T$ in the equilibrium point $\mathbf{x}_e$; $i, j = 1, \ldots, n$.

Theorem 1 (The Shilnikov Homoclinic Theorem) [Zhou & Chen, 2006, Theorem 2.2]. *Let $n = 3$, and let $\alpha, \beta \pm i\gamma$ be the eigenvalues of the matrix $D(\mathbf{x}_e)$, where $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha \cdot \beta < 0$, and $\gamma \neq 0$ (the equilibrium point is a saddle focus).*

Suppose that the following conditions are fulfilled:

(i) $|\alpha| > |\beta|$;
(ii) *there exists a homoclinic orbit connected at* $\mathbf{x}_e$.

Then:

(i) *in a neighborhood of the homoclinic orbit there is a countable number of Smale horseshoes in discrete dynamics of system (1)*;
(ii) *for any sufficiently small C^1-perturbation $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))^T$ of the function $\mathbf{f}(\mathbf{x})$ in system (1) the perturbed system $\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}) \in \mathbb{R}^n$ has at least a finite number of Smale horseshoes in the discrete dynamic defined near the homoclinic orbit*;
(iii) *both the original system (1) and the perturbed system $\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x})$ have the horseshoe type of chaos.*

Let $n = 3$, and let $\rho \pm i\omega, 0$ be the eigenvalues of the matrix $D(\mathbf{x}_e)$. In [Chen *et al.*, 2009], we see with the help of suitable transformations to reduce system (1) to the form

$$\begin{pmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{z}(t)\end{pmatrix} = \begin{pmatrix}\rho & -\omega & 0\\ \omega & \rho & 0\\ 0 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix} + \begin{pmatrix}axz + byz + o(3)\\ ayz + bxz + o(3)\\ cz^2 + o(3)\end{pmatrix}. \tag{4}$$

Theorem 2 [Chen *et al.*, 2009, Theorem 2.1]. *Let in system (4) $\rho \cdot \omega \neq 0$.*

Suppose that the following conditions are fulfilled:

(i) $c \cdot \rho < 0, c \cdot \omega > 0$;
(ii) *there exists a homoclinic orbit connected at* $\mathbf{x}_e$.

Then:

(i) *in a neighborhood of the homoclinic orbit there is a countable number of Smale horseshoes in discrete dynamics of system (4)*;
(ii) *system (4) possesses the horseshoe type of chaos.*

The stable and unstable manifolds $\mathbb{W}^s(\mathbf{e}_0)$ and $\mathbb{W}^u(\mathbf{e}_0)$ for some equilibrium point $\mathbf{e}_0$ [Kuznetsov, 1998] may be defined as

$$\mathbb{W}^s(\mathbf{e}_0) := \left\{\mathbf{x}_0 \in \mathbb{R}^n \,\middle|\, \lim_{t\to\infty}\mathbf{x}(t, \mathbf{x}_0) = \mathbf{e}_0\right\},$$

$$\mathbb{W}^u(\mathbf{e}_0) := \left\{\mathbf{x}_0 \in \mathbb{R}^n \,\middle|\, \lim_{t\to-\infty}\mathbf{x}(t, \mathbf{x}_0) = \mathbf{e}_0\right\}.$$

Let $\mathbf{A}$ be a linear operator, a matrix of which in some base of the space $\mathbb{R}^n$ coincides with the matrix A of system (1). Assume that $\mathbf{e}_0 = 0$. Denote by $\mathbb{T}_s, \mathbb{T}_u$ and $\mathbb{T}_c$ invariant with respect to the operator $\mathbf{A}$ subspaces in $\mathbb{R}^n$ such that a spectrum of the restriction of $\mathbf{A}|_{\mathbb{T}_s}$ consists of eigenvalues with negative real parts; the spectrum of the restriction of $\mathbf{A}|_{\mathbb{T}_u}$ consists of eigenvalues with positive real parts, and the spectrum of the restriction of $\mathbf{A}|_{\mathbb{T}_c}$ consists of eigenvalues with zero real parts.

Theorem 3 (The Hadamard–Perron Theorem) [Kuznetsov, 1998]. *Let $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))^T$ be a C^r-differentiable vector field with the hyperbolic equilibrium point 0 and the linear part $A\mathbf{x}$ in 0. Let ϕ^t be a flow of system (1). Then system (1) has two C^r-differentiable invariant with respect to the flow ϕ^t manifolds $\mathbb{W}^s(0)$ and $\mathbb{W}^u(0)$ passing through 0 and touching 0 at spaces $\mathbb{T}_s$ and $\mathbb{T}_u$ respectively. The solutions with initial values at $\mathbb{W}_s(0)(\mathbb{W}_u(0))$ exponentially tends to 0 at $t \to \infty (t \to -\infty)$. Besides, system (1) has third C^{r-1}-differentiable invariant with respect to the flow ϕ^t manifold $\mathbb{W}^c(0)$ passing through 0 and touching 0 at space $\mathbb{T}_c$.*

2. Triangular Systems

Consider the homogeneous system (2) of the quadratic differential equations with the vector of initial values $\mathbf{x}^T(0)$.

Note that in system (2) any quadratic form can be uniquely presented as the sum

$$\mathbf{x}^T B_{i+1}\mathbf{x} = U_{1,i+1}(x_1, \ldots, x_i) + U_{2,i+1}(x_1, \ldots, x_n),$$

where

$$U_{1,i+1}(x_1,\dots,x_i) = (x_1,\dots,x_i,0,\dots,0) \times B_{i+1}(x_1,\dots,x_i,0,\dots,0)^T,$$
$$U_{2,i+1}(x_1,\dots,x_n) = \mathbf{x}^T B_{i+1}\mathbf{x} - U_{1,i+1},$$

are quadratic forms depending on i and n variables, and $U_{11}(x_0) \equiv 0$, $U_{21}(x_1,\dots,x_n) = \mathbf{x}^T B_1 \mathbf{x}$; $i = 1,\dots,n-1$.

Let us introduce for system (2) new variable $\mathbf{y}(t) = (y_1(t),\dots,y_n(t))^T$ defined by the formula $\mathbf{x}(t) = S\mathbf{y}(t)$, where $S \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. Then, we obtain

$$\begin{pmatrix} \dot{y}_1(t) \\ \vdots \\ \dot{y}_n(t) \end{pmatrix} = S^{-1} \begin{pmatrix} (S\mathbf{y}(t))^T B_1 (S\mathbf{y}(t)) \\ \vdots \\ (S\mathbf{y}(t))^T B_n (S\mathbf{y}(t)) \end{pmatrix}. \tag{5}$$

(Thus, the vector of initial data is $\mathbf{y}(0) = S^{-1}\mathbf{x}(0)$.)

Assume that we can find an invertible matrix S such that in variables $y_1,\dots,y_n$ system (5) takes the form

$$\dot{\mathbf{y}}(t) = \begin{pmatrix} \dot{y}_1(t) \\ \vdots \\ \dot{y}_n(t) \end{pmatrix} = \mathbf{W}(\mathbf{y}(t)) = \begin{pmatrix} U_{21}(y_1(t),\dots,y_n(t)) \\ \vdots \\ U_{2n}(y_1(t),\dots,y_n(t)) \end{pmatrix}. \tag{6}$$

(We mark that the operator $\mathbf{W} : \mathbb{R}^n \to \mathbb{R}^n$ has i-dimension invariant subspaces $\mathbb{Y}_i \subset \mathbb{R}^n$ consisting of vectors $(\underbrace{*,\dots,*}_{i},0,\dots,0)^T$; $i = 1,\dots,n-1$.)

Definition 3. System (6) is called a triangular system.

For example, if $n = 2$, then (6) has the form

$$\begin{cases} \dot{y}_1(t) = a_{11}y_1^2 + 2a_{12}y_1y_2 + a_{22}y_2^2, \\ \dot{y}_2(t) = 2b_{12}y_1y_2 + b_{22}y_2^2; \end{cases} \tag{7}$$

if $n = 3$, then (6) has the form

$$\begin{cases} \dot{y}_1(t) = a_{11}y_1^2 + 2a_{12}y_1y_2 + a_{22}y_2^2 + 2a_{13}y_1y_3 \\ \qquad\quad + 2a_{23}y_2y_3 + a_{33}y_3^2, \\ \dot{y}_2(t) = 2b_{12}y_1y_2 + b_{22}y_2^2 + 2b_{13}y_1y_3 \\ \qquad\quad + 2b_{23}y_2y_3 + b_{33}y_3^2, \\ \dot{y}_3(t) = 2c_{13}y_1y_3 + 2c_{23}y_2y_3 + c_{33}y_3^2. \end{cases}$$

Now we represent the construction method of the triangulation system (6).

The last equation of system (6) has the form

$$\dot{y}_n(t) = \mathbf{y}^T P \mathbf{y} = 2c_{1n}y_1y_n + \cdots + 2c_{n-1,n}y_{n-1}y_n + c_{nn}y_n^2. \tag{8}$$

The symmetric matrix P of this equation is

$$P = \begin{pmatrix} 0 & \dots & 0 & c_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & c_{n-1,n} \\ c_{1n} & \dots & c_{n-1,n} & c_{nn} \end{pmatrix}.$$

Denote by $f_1,\dots,f_n$ elements of the last row of the matrix $\det S \cdot S^{-1}$. Then the last equation of system (5) is

$$\dot{y}_n(t) = (S\mathbf{y})^T (f_1B_1 + \cdots + f_nB_n)\frac{S\mathbf{y}}{\det S}.$$

The symmetrical matrix of this equation is $P_1 = S^T(f_1B_1 + \cdots + f_nB_n)S/\det S$. It is clear that $P = P_1$. Let $\mathbf{s}_1 = (s_{11},\dots,s_{n1})^T,\dots,\mathbf{s}_{n-1} = (s_{1,n-1},\dots,s_{n,n-1})^T$ be the first n columns of the matrix S. Using the definition of inverse matrix, we notice that $f_1,\dots,f_n$ are homogeneous polynomials of degree $n-1$ with respect to unknown scalar elements $s_{11},\dots,s_{n1},\dots,s_{1,n-1},\dots,s_{n,n-1}$, which are arranged in the first $n-1$ columns of the matrix S. Then from the equation $P = P_1$, it follows that

$$\mathbf{s}_i^T(f_1B_1 + \cdots + f_nB_n)\mathbf{s}_j = 0, \quad i,j \in \{1,\dots,n-1\},\ i < j, \tag{9}$$

where system (9) consist of $n(n-1)/2$ homogeneous equations of degree $2+n-1 = n+1$ with respect to $n(n-1)$ unknowns $s_{11},\dots,s_{n-1,n}$ under the condition $\det S \neq 0$.

Let us introduce the matrix

$$(\mathbf{s}_1, \ldots, \mathbf{s}_{n-1}) = \begin{pmatrix} I_{n-1} \\ ----- \\ v_1, \ldots, v_{n-1} \end{pmatrix} \cdot D \in \mathbb{R}^{n \times (n-1)}, \tag{10}$$

where I_{n-1} is the identity matrix of order $n-1$; the matrix $D \in \mathbb{R}^{(n-1)\times(n-1)}$ is invertible and variables $v_1, \ldots, v_{n-1}$ depend on elements $s_{ij}; i = 1, \ldots, n; j = 1, \ldots, n-1$. Then system (9) can be presented in the following form

$$\det D \cdot D^T \begin{pmatrix} & | & v_1 \\ I_{n-1} & | & \vdots \\ & | & v_{n-1} \end{pmatrix} \times (-v_1 B_1 + \cdots + (-1)^{n-1} v_{n-1} B_{n-1} + (-1)^n B_n) \begin{pmatrix} I_{n-1} \\ ----- \\ v_1, \ldots, v_{n-1} \end{pmatrix} \cdot D = 0.$$

The last system is equivalent to the system

$$N(v_1, \ldots, v_{n-1}) \equiv \begin{pmatrix} & | & v_1 \\ I_{n-1} & | & \vdots \\ & | & v_{n-1} \end{pmatrix} \times (-v_1 B_1 + \cdots + (-1)^{n-1} v_{n-1} B_{n-1} + (-1)^n B_n) \begin{pmatrix} I_{n-1} \\ ----- \\ v_1, \ldots, v_{n-1} \end{pmatrix} = 0 \tag{11}$$

consisting of $n(n-1)/2$ equations with respect to $n-1$ unknowns $v_1, \ldots, v_{n-1}$. (It is clear that by virtue of the symmetry of matrices $B_1, \ldots, B_n$ the matrix $N(v_1, \ldots, v_{n-1}) \in \mathbb{R}^{(n-1)\times(n-1)}$ is also symmetrical.) In total, we notice that system (11) has a solution only for $n = 2$. If $n > 2$, then some restrictions on matrices $B_1, \ldots, B_n$ should be realized for the solvability of this system.

Thus, the construction method of system (6) is the following.

(1) Solve the system of Eq. (11).
(2) Find matrix $(\mathbf{s}_1, \ldots, \mathbf{s}_{n-1})$ from (10), where $D \in \mathbb{R}^{(n-1)\times(n-1)}$ is any invertible matrix.
(3) Compose the matrix $S = (\mathbf{s}_1, \ldots, \mathbf{s}_{n-1}, \mathbf{s}_n) \in \mathbb{R}^{n\times n}$, where the vector $\mathbf{s}_n$ satisfies the condition $\det S \neq 0$.
(4) Build system (5).
(5) Assume in system (5) $y_n = 0$ and repeat items 1–4 for a new derived system consisting of $n-1$ equations with respect to $n-1$ unknowns $y_1, \ldots, y_{n-1}$.

We formally calculate all first derivatives with respect to time for functions $z_1 = y_1/y_n, \ldots, z_{n-1} = y_{n-1}/y_n$ of system (6). Then, we obtain

$$\begin{pmatrix} \dot{z}_1(t) \\ \vdots \\ \dot{z}_{n-1}(t) \\ \dot{y}_n(t) \end{pmatrix} = \begin{pmatrix} \dfrac{\dot{y}_1 y_n - y_1 \dot{y}_n}{y_n^2} \equiv G_1(z_1(t), \ldots, z_{n-1}(t)) y_n(t) \\ \vdots \\ \dfrac{\dot{y}_{n-1} y_n - y_{n-1} \dot{y}_n}{y_n^2} \equiv G_{n-1}(z_1(t), \ldots, z_{n-1}(t)) y_n(t) \\ G_n(z_1(t), \ldots, z_{n-1}(t)) y_n^2(t) \end{pmatrix}, \tag{12}$$

where $G_i(z_1, \ldots, z_{n-1})$ is a nonhomogeneous quadratic function and $G_n(z_1, \ldots, z_{n-1})$ is a nonhomogeneous linear function of variables $z_1, \ldots, z_{n-1}$; $i = 1, \ldots, n-1$.

Let us introduce the function z, linear with respect to $z_1, \ldots, z_{n-1}$, by the formula

$$z = 2c_{1n} z_1 + \cdots + 2c_{n-1,n} z_{n-1},$$

where $2c_{1n}, \ldots, 2c_{n-1,n}$ are the coefficients of Eq. (8).

Taking into account formula (12), we also compose the quadratic function

$$G(z_1, \ldots, z_{n-1}) \equiv 2c_{1n} G_1(z_1, \ldots, z_{n-1}) + \cdots + 2c_{n-1,n} G_{n-1}(z_1, \ldots, z_{n-1}),$$

and the quadratic form

$$h_{n-1}(\mathbf{y}) = y_n^2 G(z_1, \ldots, z_{n-1}).$$

Let $\mathbb{Y}_i$ be a linear subspace in $\mathbb{R}^n$ of dimension i, which is formed by all vectors $\mathbf{y}_i = (y_1, \dots, y_i, 0, \dots, 0)^T; i = 1, \dots, n$.

Construct the chain of inclusions: $0 = \mathbb{Y}_0 \subset \mathbb{Y}_1 \subset \cdots \subset \mathbb{Y}_{n-1} \subset \mathbb{Y}_n = \mathbb{R}^n$.

Let $\mathbf{W}_i = \mathbf{W}|_{\mathbb{Y}_i}$ be a restriction of operator $\mathbf{W}$ on the subspace $\mathbb{Y}_i$. It is easily checked that $\mathbf{W}(\mathbb{Y}_i) = \mathbb{Y}_i; i = 1, \dots, n$.

Introduce the following triangular systems:

$$\dot{\mathbf{y}}_i(t) = \begin{pmatrix} \dot{y}_1(t) \\ \vdots \\ \dot{y}_i(t) \end{pmatrix} = \mathbf{W}_i(\mathbf{y}_i(t)) = \begin{pmatrix} U_{21}(y_1(t), \dots, y_i(t), 0, \dots, 0) \\ \vdots \\ U_{2i}(y_1(t), \dots, y_i(t), 0, \dots, 0) \end{pmatrix}; \quad i = 1, \dots, n. \tag{13}$$

(It is obvious that at $i = n$, system (13) coincides with system (6).)

By analogy to system (6), we will introduce forms $h_{i-1}(\mathbf{y}_i)$ for systems (13); $i = 2, \dots, n$. (Here $h_{n-1}(\mathbf{y}_n) \equiv h_{n-1}(\mathbf{y})$.)

Theorem 4 [Belozyorov, 2011a, Theorem 3]. *Let $n > 1$. Assume that for the triangular system (6) $y_{n0} \neq 0$, and*

(i) *$\forall\, i \in \{2, \dots, n\}$ the quadratic form $h_{i-1}(\mathbf{y}_i)$ is negative definite*;

(ii) *for $i = 2$ [it will be system (7)] $a_{11}(a_{11} - 2b_{12}) < 0$.*

Then any trajectory $\mathbf{y}(t, \mathbf{y}_0)$ of system (6) is a homoclinic orbit and the equilibrium 0 is a unique globally attractive set of this system.

Suppose that there exists a linear invertible transformation $S \in \mathbb{R}^{n \times n}$ reducing system (2) to system (6). Applying the transformation S to system (1), we have

$$\dot{\mathbf{y}}(t) = \begin{pmatrix} \dot{y}_1(t) \\ \vdots \\ \dot{y}_n(t) \end{pmatrix} = D\mathbf{y}(t) + \begin{pmatrix} U_{21}(y_1(t), \dots, y_n(t)) \\ \vdots \\ U_{2n}(y_1(t), \dots, y_n(t)) \end{pmatrix}, \tag{14}$$

where $D = S^{-1}AS$ $\mathbf{y}(0) = S^{-1}\mathbf{x}(0)$.

Theorem 5 [Belozyorov, 2011a, Theorem 4]. *Let $n > 1$. Assume that for the triangular system (6) conditions (i) and (ii) of Theorem 4 hold. If either 1D space $\mathbb{Y}_1$ is not an eigenvector of the matrix D or $\mathbf{y}(0) \notin \mathbb{Y}_1$, then for any initial values, all solutions of system (14) are bounded.*

3. Case of Singular Linear Part in System (1)

Let $n = 3$. We will take for simplicity that $y_1 = x$, $y_2 = y$, $y_3 = z$. Suppose that in these variables, system (14) has

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = D \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 \\ 2b_{12}xy + b_{22}y^2 + 2b_{13}xz + 2b_{23}yz + b_{33}z^2 \\ 2c_{13}xz + 2c_{23}yz + c_{33}z^2 \end{pmatrix}, \tag{15}$$

where $\lambda_1 = \mu, \lambda_{2,3} = \rho \pm i\omega, i = \sqrt{-1}$ are eigenvalues of the matrix D and $\rho \cdot \omega \neq 0$, $\mu \cdot \rho \leq 0$ (we suppose for definiteness $\mu \leq 0$ and $\rho > 0$).

Introduce the quadratic forms:

$$f(x, y) = b_{12}(a_{11} - 2b_{12})x^2 + b_{12}(2a_{12} - b_{22})xy + b_{12}a_{22}y^2$$

and

$$\begin{aligned} g(x, y, z) &= c_{13}(a_{11} - 2c_{13})x^2 + 2(c_{13}a_{12} + c_{23}b_{12} - 2c_{13}c_{23})xy + (c_{13}a_{22} + c_{23}b_{22} - 2c_{23}^2)y^2 \\ &\quad + (2c_{13}a_{13} + 2c_{23}b_{13} - c_{13}c_{33})xz \\ &\quad + (2c_{13}a_{23} + 2c_{23}b_{23} - c_{23}c_{33})yz \\ &\quad + (c_{13}a_{33} + c_{23}b_{33})z^2. \end{aligned}$$

Theorem 6. *Let the following conditions be valid for system (15):*

(i) *the quadratic forms $f(x,y)$ and $g(x,y,z)$ are negative definite;*
(ii) $a_{11}(a_{11}-2b_{12})<0$;
(iii) *the vector $(\alpha,0,0)^T,\alpha\neq 0$, is not an eigenvector of the matrix D.*

Then in system (15) either there is a limit cycle or a limit torus or a complex irregular dynamics takes place.

Proof. Note that the conditions (i)–(iii) of Theorem 6 are the conditions of Theorem 5 guaranteeing boundedness of the solutions of system (15).

(1) With the help of suitable real linear transformations, we reduce system (15) to the form

$$\begin{pmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{z}(t)\end{pmatrix}=\begin{pmatrix}p_{11} & p_{12} & 0\\ p_{21} & p_{22} & 0\\ 0 & 0 & \mu\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix}+\begin{pmatrix}f_1(x,y,z)\\ f_2(x,y,z)\\ f_3(x,y,z)\end{pmatrix},\tag{16}$$

where for simplicity we left old designations of the new variables x,y,z; $f_1(x,y,z),f_2(x,y,z),f_3(x,y,z)$ are quadratic forms.

(2) Let us calculate the Lyapunov's exponent Λ for a real function $f(t)$ in accordance with the known formula [Belozyorov, 2011b]:

$$\Lambda[f]=\overline{\lim_{t\to\infty}}\frac{1}{t}\ln\left|\frac{f(t)}{f(t_0)}\right|.\tag{17}$$

Then by virtue of boundedness of the solutions $x(t),y(t),z(t)$ of system (16) for any x_0,y_0, and any z_0 (see Theorem 5), we get $\Lambda[x(t)]\le 0$, $\Lambda[y(t)]\le 0$, and $\Lambda[z(t)]\le 0$.

We take advantage of the following properties of Lyapunov's exponents:

(c1) if the function $f_1(t)$ has a strict Lyapunov's exponent $\Lambda[f_1(t)]$, then $\Lambda[f_1(t)\cdot f_2(t)]=\Lambda[f_1(t)]+\Lambda[f_2(t)]$;
(c2) if $m\ge 0$, then $\Lambda[t^m]=0$;
(c3) if $\Lambda[f(t)]<0$, then $\Lambda[\int_t^\infty f(\tau)d\tau]\le\Lambda[f(t)]$;
(c4) $\Lambda[f_1(t)+f_2(t)]\le\max(\Lambda[f_1(t)],\Lambda[f_2(t)])$;
(c5) $\Lambda[d\cdot f(t)]=\Lambda[f(t)](d\neq 0)$.

We write the last equation of system (16) in the integral form

$$\begin{aligned}z(t)&=z_0\exp(\mu t)\\&\quad+\int_0^t\exp(\mu(t-\tau))\cdot f_3(x(\tau),y(\tau),z(\tau))d\tau\\&\equiv z_1(t)+z_2(t),\quad (t>\tau),\end{aligned}\tag{18}$$

where $z_1(t)=z_0\exp(\mu t)$, $z_2(t)=z(t)-z_1(t)$. Then from properties (c1)–(c5) and boundedness of the solutions $x(t),y(t),z(t)$ it follows that at $\mu=0$ we have $\Lambda[z(t)]=0$.

Let $\mu<0$. Then from (c5) it follows that $\Lambda[z_1(t)]=\mu$. From (c1) and (c3) we have

$$\begin{aligned}&\Lambda[\exp(\mu(t-\tau))\cdot f_3(x(\tau),y(\tau),z(\tau))]\\&\quad=\Lambda[\exp\mu t]+\Lambda[f_3(x(\tau),y(\tau),z(\tau))]\\&\quad=\mu+0=\mu\end{aligned}$$

and, therefore, $\Lambda[z_2(t)]\le\mu+\delta$, where $\delta\le 0$. Thus, from (c4) we have $\Lambda[z(t)]=\Lambda[z_1(t)+z_2(t)]=\max(\mu,\mu+\delta)=\mu<0$.

Under the conditions of Theorem 6 for any eigenvalues of the Jacobi matrix at any equilibrium, Lyapunov's exponents of system (15) are defined (to within permutations) by four possibilities: $(0,0,0)$; $(-,0,0)$; $(-,-,0)$; $(-,-,-)$.

Trajectories, for which Lyapunov's exponents $(\Lambda[x(t)],\Lambda[y(t)],\Lambda[z(t)])$ adopt one of the values $(-,-,0)$, $(-,0,-)$ or $(0,-,-)$, are limit cycles. Trajectories, for which Lyapunov's exponents $(\Lambda[x(t)],\Lambda[y(t)],\Lambda[z(t)])$ adopt one of the values $(-,0,0)$, $(0,-,0)$ or $(0,0,-)$, are limit tori. If $(\Lambda[x(t)],\Lambda[y(t)],\Lambda[z(t)])=(0,0,0)$ then in system (15) a complex irregular (chaotic) dynamics can arise. ∎

Theorem 7. *Assume that for system (15) all conditions of Theorem 6 are fulfilled. We also suppose that all nonzero equilibrium points of this system are either by saddle nodes or saddle focuses (including singular equilibrium points). Then in the system there exist homoclinic or heteroclinic orbits.*

Proof

(1) Let $\mathbb{W}_u(0)$ be 2D unstable and $\mathbb{W}_c(0)$ be 1D central manifolds of point $O(0,0,0)$ (if $\mu\neq 0$, then $\mathbb{W}_c(0)$ must be replaced by 1D stable manifold $\mathbb{W}_s(0)$). Then, by virtue of boundedness of

the solutions of system (15), the part trajectories $\mathbb{S}_1 \subset \mathbb{W}_u(0)$ ($\mathbb{W}_u(0) - \mathbb{S}_1 = \mathbb{S}_2 \neq \emptyset$) are attracted to some equilibrium points of system (15); among them, it can be the equilibrium point $(0,0,0)$. It means the existence of heteroclinic or homoclinic orbits.

(2) Assume that any trajectory of the set $\mathbb{S}_2 \subset \mathbb{W}_u(0)$ such that $\mathbb{S}_1 \cap \mathbb{S}_2 = \emptyset$ is not attracted to any nonzero equilibrium points. Then, by virtue of boundedness of solutions of system (15), the trajectories of the set $\mathbb{S}_2 \subset \mathbb{W}_u(0)$ must be attracted to some limit attractive set (cycle or torus) L.

(3) Suppose $\mu = 0$. It is known that in a small neighborhood of origin the solutions of system (15) have the form: $x(t) = \exp(\rho t)(x_0 \cos(\omega t) - y_0 \sin(\omega t))$, $y(t) = \exp(\rho t)(x_0 \sin(\omega t) + y_0 \cos \times (\omega t))$, $z(t) = z_0/(1 - c_{33} z_0 t)$. Let $c_{33} z_0 < 0$. Then the central manifolds $\mathbb{W}_c(0)$ near the point O is topologically equivalent to a stable manifold. We change the sign of time to opposite. Then the attractor L becomes unstable, but the trajectory $\mathbb{W}_c(0)$ is attracted to this attractor, which is a contradiction. Consequently, $\mathbb{W}_u(0) \cap \mathbb{W}_c(0) \neq 0$ and the trajectory $\mathbb{W}_c(0)$ is attracted to point O. Thus, in system (15) there exists a homoclinic orbit connected at point O.

(4) Let $\mu \neq 0$. Then the solution $z(t) = z_0/(1 - c_{33} z_0 t)$ must be replaced by the solution $z(t) = z_0 \exp(\mu t)$. Now we repeat all reasonings of item 3. ■

4. Existence of Chaotic Dynamics in System (15)

Let us consider that in system (15) the matrix D has the form:

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ 0 & 0 & d_{33} \end{pmatrix}. \tag{19}$$

Let $\rho \mp i\omega$, d_{33} be a spectrum of the matrix D. We will consider that $\rho > 0, \omega > 0, d_{33} \leq 0$. Denote by x_0, y_0 and z_0 the initial values for system (15). Besides, we suppose that $x_0 = y_0 = 0$. Transform the expression $c_{13}x(t) + c_{23}y(t)$ in the following way:

$$\begin{aligned} &c_{13}x(t) + c_{23}y(t) \\ &\quad = z(t)\left(c_{13}\frac{x(t)}{z(t)} + c_{23}\frac{y(t)}{z(t)}\right) \\ &\quad = z(t)\int_{t_0}^{t}\left[c_{13}\left(\frac{\dot{x}(\tau)}{z(\tau)}\right) + c_{23}\left(\frac{\dot{y}(\tau)}{z(\tau)}\right)\right]d\tau \\ &\quad = z(t)\int_{t_0}^{t}\left[\frac{h(x(\tau),y(\tau),z(\tau))}{z^2(\tau)}\right]d\tau \\ &\qquad + z(t)\int_{t_0}^{t} z(\tau)\left[\frac{g(x(\tau),y(\tau),z(\tau))}{z^2(\tau)}\right]d\tau, \end{aligned} \tag{20}$$

where

$$\begin{aligned} h(x,y,z) &= 2[(d_{11} - d_{33})c_{13} + d_{21}c_{23}]xz \\ &\quad + 2[d_{12}c_{13} + (d_{22} - d_{33})c_{23}]yz \\ &\quad + 2[d_{13}c_{13} + d_{23}c_{23}]z^2 \end{aligned}$$

and $g(x,y,z)$ as in Sec. 3 is a quadratic form of variables x, y, z.

We note the symmetric matrices H and $z \cdot G$ of the forms $h(x,y,z)$ and $zg(x,y,z)$ as:

$$H = \begin{pmatrix} 0 & 0 & h_1 \\ 0 & 0 & h_2 \\ h_1 & h_2 & h_3 \end{pmatrix}, \quad z \cdot G = z \cdot \begin{pmatrix} g_1 & g_2 & g_3 \\ g_2 & g_4 & g_5 \\ g_3 & g_5 & g_6 \end{pmatrix}.$$

Let $z > 0$. Note the conditions of a negative definiteness of the matrix $H + z \cdot G$. It is simple to check that by virtue of structure matrix H, the first two conditions ($g_1 < 0$, $g_1 g_4 - g_2^2 > 0$) coincide with the first two conditions of negative definiteness of the matrix G. Third condition has the form:

$$\begin{aligned} &\det(H + z \cdot G) \\ &\quad = z[(\det G)z^2 + v_1 z + v_2] \\ &\quad \equiv z[(\det G)z^2 + (g_1 g_4 h_3 - 2g_1 g_5 h_2 + 2g_2 g_5 h_1 \\ &\qquad - g_2^2 h_3 + 2g_2 g_3 h_2 - 2g_3 g_4 h_1)z \\ &\qquad + (-g_4 h_1^2 + 2g_2 h_1 h_2 - g_1 h_2^2)] < 0. \end{aligned}$$

By virtue of the negative definiteness of matrix G, we have $\det G < 0$. Therefore, in order that the condition $\det(H + z \cdot G) < 0$ is met, it is necessary and sufficient that discriminant $\mathrm{Disc} \equiv v_1^2 - 4v_2 \det G$ of the quadratic polynomial $(\det G)z^2 + v_1 z + v_2$ is negative.

The function $c_{13}x(t) + c_{23}y(t)$ explicitly does not depend on $z(t)$. Therefore, if $z \leq 0$, then the condition $\mathrm{Disc} < 0$ is again necessary and sufficient in order that $(\det G)z^2 + v_1 z + v_2 < 0$.

Theorem 7 does not give an answer for the following question: What orbits (homoclinic or heteroclinic) exist for the concrete system (15)? Theorem 8 removes this omission.

Theorem 8. *Assume that for system (15) all conditions of Theorem 7 are valid. We also suppose that the matrix D has the form (19). By (x_i^*, y_i^*, z_i^*) denote all equilibrium points of system (15) for which $z_i^* = 0$; $i = 1, \dots, m$. Suppose that eigenvalues of the Jacobian matrix of system (15) at point $(x_i^*, y_i^*, 0)$ equal $a_i \pm b_i\sqrt{-1}, c_i$, where $a_i \neq 0$, $a_i c_i \leq 0$. Then $\forall\, i \in \{1, \dots, m\}$ there exists a homoclinic orbit connected at point $(x_i^*, y_i^*, 0)$ which is situated either in half-space $z \geq 0$ or half-space $z \leq 0$.*

Proof. Find the solution $z(t)$ of the third equation of system (15) with matrix D (19):

$$\dot{z}(t) = c_{33}z^2(t) + [c_{13}x(t) + c_{23}y(t) + d_{33}]z(t). \tag{21}$$

This solution has the form

$$z(t) = \frac{z_0 \exp(q(t))}{1 - c_{33}z_0 \displaystyle\int_{t_0}^{t} \exp(q(\tau))d\tau}, \tag{22}$$

where $q(t) = \int_{t_0}^{t}[c_{13}x(\tau) + c_{23}y(\tau) + d_{33}]d\tau$ and $\forall\, t > 0$ $\int_{t_0}^{t} \exp(q(\tau))d\tau > 0$.

From (22), it follows that the existence of the homoclinic orbit may be ensured by the condition $\lim_{t\to\infty} c_{13}x(t) + c_{23}y(t) \leq 0$. In order that this condition was realized, it is sufficient that $\det(H + z \cdot G) < 0$. The last inequality can be confirmed by the obvious equality:

$$\lim_{z\to\infty} \det(H + z \cdot G)$$
$$= \lim_{z\to\infty} z[(\det G)z^2 + v_1 z + v_2] = -\infty.$$

Since $d_{33} \leq 0$, then from (20) and (22), it follows that

$$0 \leq \lim_{t\to\infty} \int_{t_0}^{t} \exp(q(\tau))d\tau < \infty$$

and therefore, $\lim_{t\to\infty} z(t) = 0$.

Let $\mathbb{W}^s(x_i^*, y_i^*, 0)$ $(\mathbb{W}^u(x_i^*, y_i^*, 0))$ be a stable (unstable) manifold of the point $(x_i^*, y_i^*, 0)$; $i \in 1, \dots, m$. Assume that initial values $(x_0, y_0, z_0) \in \mathbb{W}^u(x_i^*, y_i^*, 0)$. We derived the following result: $\lim_{t\to\infty}(x(t), y(t), z(t)) = (x_i^*, y_i^*, 0)$.

Now we change $t \to -t$. According to Theorem 5, all solutions of system (15) will remain bounded. Then, we have $\lim_{t\to-\infty}(x(t), y(t), z(t)) = (x_i^*, y_i^*, 0)$. Therefore, $(x_0, y_0, z_0) \in \mathbb{W}^s(x_i^*, y_i^*, 0)$. It means the existence of the homoclinic orbit connected at $(x_i^*, y_i^*, 0)$; $i = 1, \dots, m$.

Let $d_{33} = 0$. Then for the proof of the existence of homoclinic orbits, it is possible to use item 3 of Theorem 7. In this case near the equilibrium $(x_i^*, y_i^*, 0)$ the variety $\mathbb{W}^s(x_i^*, y_i^*, 0)$ (or $\mathbb{W}^u(x_i^*, y_i^*, 0)$) must be replaced by the central variety $\mathbb{W}^c(x_i^*, y_i^*, 0)$, $i \in \{1, \dots, m\}$.

The location of homoclinic orbits is determined by Eq. (21) and Lemma 1 [Belozyorov, 2011a]. ■

Theorem 1 or 2 guarantees the existence of chaotic dynamics in system (1) at $n = 3$. It is easily checked that condition (i) of Theorem 2 can be fulfilled by replacements of variables $(x, y, z) \to (x, \pm y, \pm z)$. Then Theorem 8 jointly with Theorem 1 or 2 allows to consider the construction method of a discrete map generating chaotic dynamics in system (15).

Under the conditions of Theorem 8, system (15) has to have a homoclinic orbit.

Introduce Poincare's section P defined by equation $c_{13}x + c_{23}y = 0$. Denote by t_0, an initial moment such that $c_{13}x(t_0) + c_{23}y(t_0) = 0$ $((x(t_0), y(t_0), z(t_0)) \in P)$. Let $t_1 = t_0 + T_0$ be a next moment of the trajectory $(x(t), y(t), z(t))$ passing by plane P. Then from (22) it follows that

$$z(t_0 + T_0) = \frac{z(t_0)\exp(q(t_0 + T_0))}{1 - c_{33}z(t_0) \displaystyle\int_{t_0}^{t_0+T_0} \exp(q(\tau))d\tau}. \tag{23}$$

In [Belozyorov, 2011a] it is shown that functions $x(t)$, $y(t)$, $z(t)$ and $x(t)/z(t)$, $y(t)/z(t)$ are bounded. Therefore, by virtue of $\forall\, t > 0$ $z(t) \neq 0$, we can introduce constants

$$\delta_0 = \int_{t_0}^{t_0+T_0} \left[\frac{h(x(\tau), y(\tau), z(\tau))}{z^2(\tau)}\right] d\tau,$$

$$\nu_0 = \int_{t_0}^{t_0+T_0} \left[\frac{g(x(\tau), y(\tau), z(\tau))}{z^2(\tau)}\right] d\tau < 0.$$

Taking into account the known theorem about the mean value of a definite integral from (20) we get

$$c_{13}x(t_0+T_0)+c_{23}y(t_0+T_0) = \delta_0 z(t_0+T_0)+\nu_0 z(t_0+T_0)\cdot z(\tau),$$

where $\tau \in (t_0, t_0+T_0)$.

Assume that in Theorem 8 $b_i \neq 0$; $i=1,\ldots,m$. Then functions $x(t)$ and $y(t)$ will oscillate in the neighborhood of equilibrium $(x_i^*, y_i^*, 0)$; $i=1,\ldots,m$. Therefore, the function $c_{13}x(t)+c_{23}y(t)$ (and it means function $z(t)$) will also oscillate. Hence, we can introduce the designations: $t_1=t_0+T_0,\ldots,t_k=t_{k-1}+T_{k-1}$, $z_k=z(t_k)$, and

$$\delta_k = \int_{t_k}^{t_k+T_k}\left[\frac{h(x(\tau),y(\tau),z(\tau))}{z^2(\tau)}\right]d\tau,$$

$$\nu_k = \int_{t_k}^{t_k+T_k}\left[\frac{g(x(\tau),y(\tau),z(\tau))}{z^2(\tau)}\right]d\tau < 0, \quad k=1,2,\ldots.$$

Then there exists a neighborhood of equilibrium point $(x^*, y^*, 0)$ in which formula (23) may be presented in the form

$$z_{k+1}=\frac{z_k\exp(d_{33}T_k+\delta_kT_kz_k+\nu_kT_kz_k^2)}{1-c_{33}z_k\displaystyle\int_0^{T_k}\exp(q(\xi))d\xi}, \quad \nu_k<0.$$

(Here the denominator is a positive bounded magnitude.)

Assume $\sqrt{-\nu_kT_k}z_k = w_k, p_k=\delta_kT_k/\sqrt{-\nu_kT_k}$, and

$$\gamma_k = -\frac{c_{33}\displaystyle\int_0^{T_k}\exp(q(\xi))d\xi}{\sqrt{-\nu_kT_k}},$$

$$r_k=\frac{\exp(d_{33}T_k)}{\sqrt{-\nu_kT_k}}, \quad k=0,1,\ldots.$$

In some neighborhood of the homoclinic orbit, we can consider that $T_k \approx T>0$, $r_k\approx r>0$, $\gamma_k\approx\gamma$, and $p_k\approx p$. Then, we derive the new discrete model

$$w_{k+1}=\alpha(w_k)\equiv\frac{rw_k\exp(pw_k-w_k^2)}{1+\gamma w_k}, \quad k=0,1,\ldots, \tag{24}$$

which describes the chaotic behavior of systems (15), at the defined values $r>0$, $\gamma\in(-d,\infty)$, and $p\in\mathbb{R}$. The number $d>0$ must be chosen so that for any integer non-negative k, $1+\gamma w_k>0$. (It is easily checked that by virtue of the multiplier $\exp(-w_k^2)$ in formula (24) the positive sequence $w_0, w_1, w_2,\ldots$ is bounded.)

Theorem 9. *If $p\geq 0$ and $\gamma\geq 0$, then there exists $r>0$ such that the discrete map (24) is chaotic.*

Proof. Introduce the 1D real map

$$\alpha(v)=\frac{rv\exp(pv-v^2)}{1+\gamma v}, \quad v\in\mathbb{V}=[0,\infty).$$

Let $r>0, p>0$, and $\gamma>0$. Then roots of the equation $\alpha'(v)=0$ are determined by the equation $2\gamma v^3+(2-p\gamma)v^2-pv-1=0$. The known Theorem of Descartes about the number of positive roots of polynomial asserts that there exists only one positive root v^* of this equation. Since $\alpha(v)\geq 0$ on the interval $[0,\infty)$, and $\alpha'(v^*-\delta)>0$, and $\alpha'(v^*+\delta)<0$ for a small $\delta>0$, then the root v^* is a unique maximum of the function $\alpha(v)$ on this interval. Thus, the function $\alpha(v)$ will be nonmonotone and unimodal on the interval $[0,\infty)$: the interval $[0,v^*)$ is an increasing interval and the interval (v^*,∞) is a decreasing interval.

By definition, take $\mathbb{W}=[0,1]$. Let $\mathbf{T}:\mathbb{V}\to\mathbb{W}$ be a continuous map given by the formula $w=(2/\pi)\cdot\arctan v$. Since $\lim_{v\to\infty}(2/\pi)\cdot\arctan v=1$, then we can consider that $\mathbf{T}$ is a homeomorphism and $\mathbf{T}(\mathbb{V})=\mathbb{W}$, $\mathbf{T}^{-1}(\mathbb{W})=\mathbb{V}$.

By

$$\phi(w)=\mathbf{T}^{-1}(\alpha(\mathbf{T}(w))) \equiv \frac{2}{\pi}\arctan\left(\tan\frac{\pi w}{2}\cdot\frac{r}{1+\gamma\tan\frac{\pi w}{2}} \cdot\exp\left(p\cdot\tan\frac{\pi w}{2}-\tan^2\frac{\pi w}{2}\right)\right) \tag{25}$$

define the continuous conjugate to α mapping $\phi:\mathbb{W}\to\mathbb{W}$ [Crownover, 1995].

(a) *Density of periodic points.* It is clear that the inverse mapping $\phi^{-1}(w)$ has two branches: $\phi_1^{-1}(w)$ and $\phi_2^{-1}(w)$, where each of the mappings $\phi_1^{-1}(w)$ and $\phi_2^{-1}(w)$ is invertible.

Define the function $\Phi:\mathbb{W}\to\mathbb{W}$ by the rule

$$\Phi(w)=\phi_{i_1}^{-1}(\phi_{i_2}^{-1}(\ldots(\phi_{i_k}^{-1}(w)))), \quad k=2,3,\ldots$$

where either $i_k=1$ or $i_k=2$. It is clear that the mapping $\Phi(w)$ is monotonical.

The mapping $\Phi(w)$ has at least one fixed point w^*. It is known that the fixed point w^* of the mapping $\Phi(w)$ corresponds to a fixed point w^{**} of the mapping $\phi^{(k)}(w) = \phi(\phi(\dots\phi(w)))$. The point w^{**} of the mapping $\phi^{(k)}(w)$ corresponds to either a fixed point or a m-cycle of the mapping $\phi(w)$. If w^{**} is a fixed point of $\phi(w)$ then it is possible only at $i_1 = i_2$. Since it is possible to take any integer k and arbitrarily choose numbers i_1, i_2 from 1 to 2, then from the condition $\phi_1^{-1}(\phi_2^{-1}(w)) \neq \phi_2^{-1}(\phi_1^{-1}(w))$ it follows that the nonmonotone function $\phi(w)$ can have any number of cycles of different multiples and an uncountable set of nonperiodic trajectories.

According to Sharkovsky's Theorem [Crownover, 1995] a cycle of period 3 implies cycles of all periods. Thus, in system $w_{i+1} = \phi(w_i)$ there exist all cycles with period $2^i, i = 0, 1, 2, \dots$. According to Singer's Theorem [Crownover, 1995] in the discrete system $w_{i+1} = \phi(w_i), i = 0, 1, 2, \dots$, at any n and some values of parameters $r = r_n$, $p = p_n$, $\gamma = \gamma_n$ there are unstable cycles of period $2^i, i = 0, \dots, n-1$, and one stable cycle of period 2^n. If $r = r_\infty$, $p = p_\infty$, and $\gamma = \gamma_\infty$, where r_∞, p_∞, and γ_∞ are parameters at which there are 3-cycles in process (24), then mapping $\phi(w)$ has a semi-stable trajectory $\mathbb{S}$ in any neighborhood of any point of this trajectory, where lie points of a countable set of unstable cycles of all periods $2^i, i = 0, 1, 2, \dots$. Therefore, a set of all periodic points is density in $\mathbb{W}$. (Any point of $\mathbb{S}$ with given accuracy can be approximated by some periodic point.)

(b) *Transitivity*. In [Belozyorov, 2012] it is shown that the function $\alpha_0(v) = rv\exp(-v)$, $v \in [0, \infty)$, is chaotic. Introduce the homeomorphism $\mathbf{H} : \mathbb{V} \to \mathbb{V}$ under the formula $\mathbf{H}(v) = 2v^2$.

Define the function $q : \mathbb{V} \to \mathbb{V}$ by the formula

$$\begin{aligned} q(v) &= \mathbf{H}^{-1}(\alpha_0(\mathbf{H}(v))) \\ &= \sqrt{r}v \cdot \exp(-v^2). \end{aligned}$$

It is clear that functions $\alpha_0(v)$ and $q(v)$ are continuous conjugate with respect to the map $\mathbf{H}$. Thus, if the function $\alpha_0(v)$ is chaotic, then the function $q(v)$ is also chaotic.

It is obvious that $\forall\, v \in \mathbb{V}$, we have

$$\begin{aligned} \alpha(v) &= rv\exp(-v^2) \cdot \frac{\exp(pv)}{1 + \gamma v} \\ &\equiv rv\exp(-v^2) \cdot \lambda(v), \end{aligned}$$

where $\lambda > 1$ at $p > 0$, $\gamma > 0$, and $v > 0$. Besides, the function $rv\exp(-v^2)$ is chaotic for some $r > 0$. From here it follows that the function $rv\exp(-v^2)$ is transitive.

Represent function (25) in the form:

$$\begin{aligned} \phi(w) = \frac{2}{\pi} \arctan \Big(r \tan \frac{\pi w}{2} \cdot \mu(w) \\ \cdot \exp\Big(-\tan^2 \frac{\pi w}{2} \Big) \Big), \end{aligned}$$

where

$$\mu(w) = \frac{\exp\Big(p \cdot \tan \dfrac{\pi w}{2}\Big)}{1 + \gamma \tan \dfrac{\pi w}{2}} > 1$$

at $p > 0, \gamma > 0$, and $\forall\, w \in \mathbb{W}$.

Consider the function

$$\begin{aligned} \phi_0(w) &= \frac{2}{\pi} \arctan \Big(r \tan \frac{\pi w}{2} \cdot \exp\Big(-\tan^2 \frac{\pi w}{2} \Big) \Big) \\ &\equiv \frac{2}{\pi} \arctan(r \cdot \eta(w)). \end{aligned}$$

(The function $\phi_0(w)$ is the function $\phi(w)$ at $p = 0$ and $\gamma = 0$.)

By virtue of transitivity of the function $rv\exp(-v^2)$ the function $\phi_0(w)$ will be also transitive. It means that for any open sets $\mathbb{U}_1, \mathbb{U}_2 \subset \mathbb{W}$ there exists a natural number n such that $\phi_0^{(n)}(\mathbb{U}_1) \cap \mathbb{U}_2 \neq \emptyset$ [Crownover, 1995].

By $d(a, b) = |a - b|$, $\forall\, a, b \in \mathbb{W}$, denote a metric on the space $\mathbb{W}$. It is clear that $\mathbb{W}$ is a complete metric space. Then, it is possible to show that the transitivity of $\phi_0(w)$ is equivalent to the condition: for any open set $\mathbb{U} \subset \mathbb{W}$, $\overline{\bigcup_{n=0}^{\infty} \phi_0^{(n)}(\mathbb{U})} = \mathbb{W}$, where $\overline{\mathbb{A}}$ is the closure of $\mathbb{A}$. Since $\forall\, w \in \mathbb{W}$ we have $\phi_0(w) \leq \phi(w)$, then from $\mu > 1$ it follows that $\forall\, w_i, w_j \in \mathbb{W}$

$$\begin{aligned} &d(\phi_0(w_i), \phi_0(w_j)) \\ &\quad = \frac{2}{\pi} |\arctan(r \cdot \eta(w_i)) - \arctan(r \cdot \eta(w_j))| \\ &\quad = \frac{2}{\pi} \left| \arctan \frac{r(\eta(w_i) - \eta(w_j))}{1 + \eta(w_i)\eta(w_j)} \right| \\ &\quad < \frac{2}{\pi} \left| \arctan \frac{r \cdot (\mu(w_i)\eta(w_i) - \mu(w_j)\eta(w_j))}{1 + \mu(w_i)\mu(w_j)\eta(w_i)\eta(w_j)} \right| \\ &\quad = d(\phi(w_i), \phi(w_j)). \end{aligned}$$

Therefore, $\forall\, w_i, w_j \in \mathbb{U}$, and any integer $n \geq 0$, we have

$$\sum_{i=0}^{n} d(\phi_0^{(i)}(w_i), \phi_0^{(i)}(w_j))$$
$$< \sum_{i=0}^{n} d(\phi^{(i)}(w_i), \phi^{(i)}(w_j)).$$

Hence, $\overline{\bigcup_{n=0}^{\infty} \phi^{(n)}(\mathbb{U})} = \mathbb{W}$ and the function $\phi(w)$ (it means that and the function $\alpha(v)$) is transitive.

Finally, from (a) and (b) it follows that for some r, p, and γ the function $\alpha(v)$ is chaotic [Crownover, 1995]. ■

5. Examples

(1) Consider the system containing a limit cycle and a homoclinic orbit (see Figs. 1 and 2):

$$\begin{cases} \dot{x}(t) = 2x - 16z - 3x^2 - xy + xz \\ \qquad\quad + y^2 - z^2, \\ \dot{y}(t) = -7xy + yz + 2.5y^2 - 2.5z^2, \\ \dot{z}(t) = 16x + 2z + z^2 - 7xz + 3yz. \end{cases} \tag{26}$$

Here the eigenvalues of Jacobi matrix at the point $O = (0, 0, 0)$ are $\lambda_{1,2} = 2 \pm 16i$, $\lambda_3 = 0$. For system (26) all conditions of Theorems 6 and 7 are valid.

(2) Consider the system also containing a homoclinic orbit (see Figs. 3 and 4):

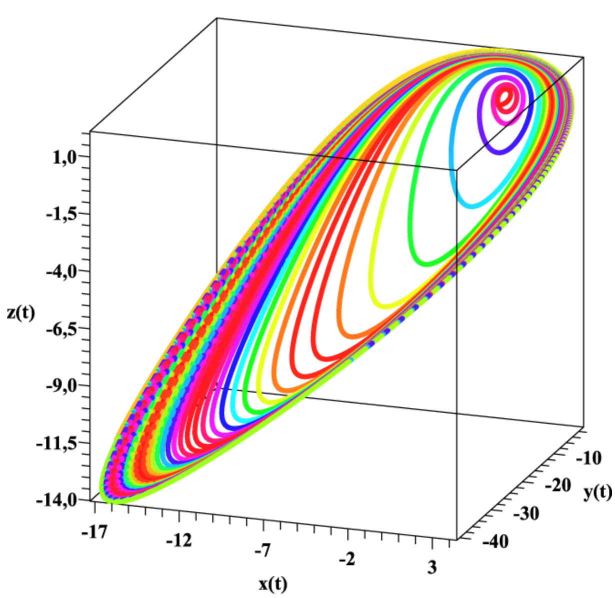

Fig. 1. The phase portrait of systems (26) with initial values $(0, -3, 0)$. There is a limit cycle.

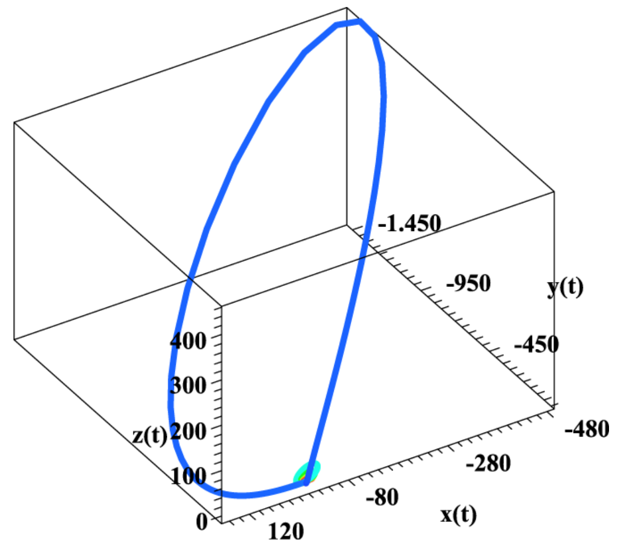

Fig. 2. The phase portrait of system (26) with initial values $(0, 0.4, 0)$. There is a homoclinic orbit connected at the point $(0; 0; 0)$.

$$\begin{cases} \dot{x}(t) = 2x - 9y + 12z - 3x^2 - xy + xz \\ \qquad\quad + y^2 - z^2, \\ \dot{y}(t) = 9x + 2y - 13z - 7xy + yz \\ \qquad\quad + 2.5y^2 - 2.5z^2, \\ \dot{z}(t) = z^2 - 7xz + 3yz. \end{cases} \tag{27}$$

For system (27) all conditions of Theorem 8 are valid.

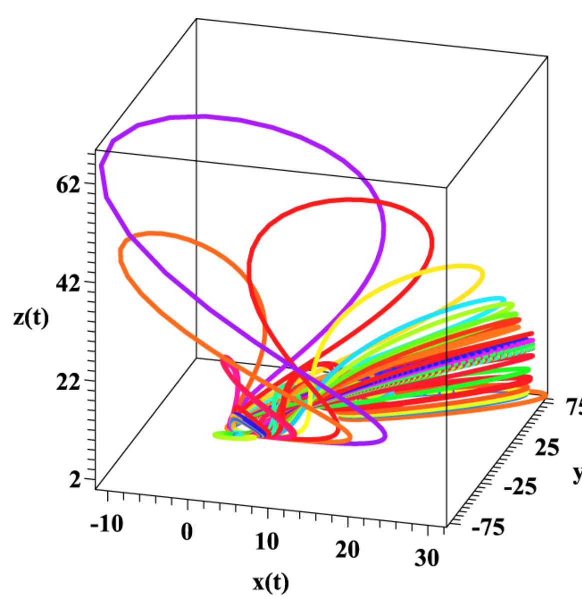

Fig. 3. Phase portrait of system (27) with initial values $(0, 0, 0.01)$. There is a homoclinic orbit connected at the point $(0; 0; 0)$ and the chaotic attractor of homoclinic type.

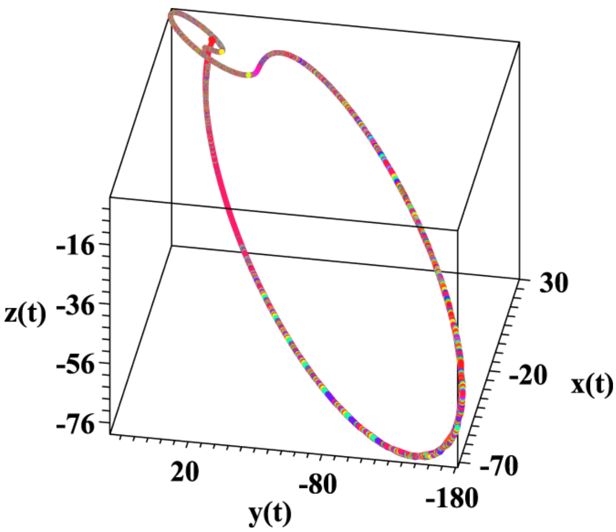

Fig. 4. The homoclinic orbit of system (27) connected at the point $(15.47; 41.15; 0)$.

(3) Another system with a chaotic attractor of homoclinic type is (see Fig. 5):

$$\begin{cases} \dot{x}(t) = 2x - 20z + 3x^2 - 2y^2 - 2xz \\ \qquad\quad - 2yz - z^2, \\ \dot{y}(t) = -x + 8xy + 4yz + 4xz + 4y^2 + z^2, \\ \dot{z}(t) = 20x + 2z + z^2 + 4xz + 2yz. \end{cases} \tag{28}$$

Initial values are $(0, -1, 0)$. There is a chaotic attractor of homoclinic type.

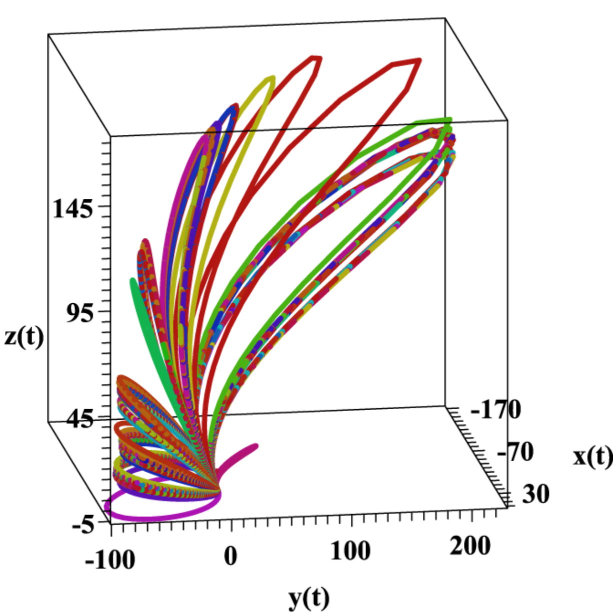

Fig. 5. The phase portrait of system (28).

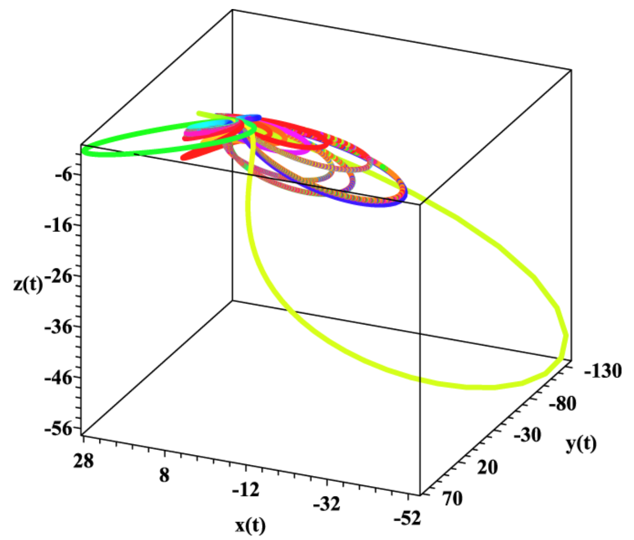

Fig. 6. The phase portrait of system (29).

(4) Consider the system also containing a chaotic attractor (see Fig. 6):

$$\begin{cases} \dot{x}(t) = 20x - 10y + 9z - 3x^2 - xy + xz \\ \qquad\quad + y^2 - z^2, \\ \dot{y}(t) = 37x - 12y + 19z - 7xy + yz \\ \qquad\quad + 2.5y^2 - 2.5z^2, \\ \dot{z}(t) = z^2 - 7xz + 3yz. \end{cases} \tag{29}$$

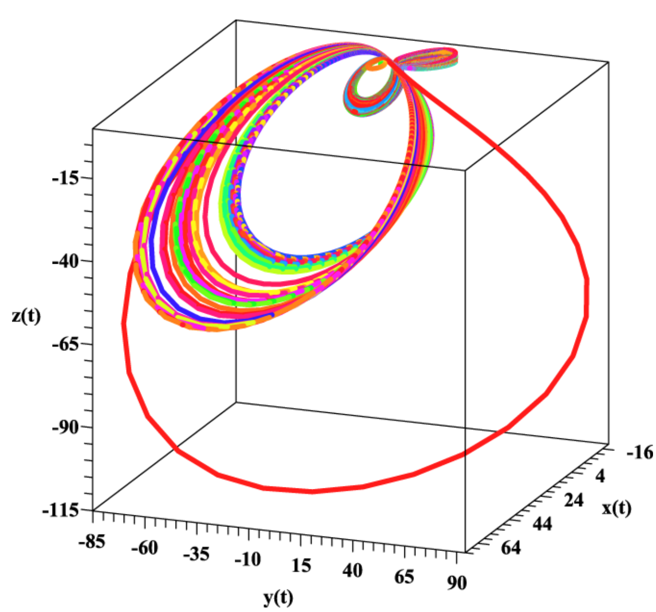

Fig. 7. The phase portrait of system (30).

Eigenvalues of the Jacobi matrix at the point $(0,0,0)$ are $\lambda_{1,2} = 4 \pm 10.677i$, $\lambda_3 = 0$; at the equilibrium point $(4.8857, 12.8551, 0)$ are $\lambda_{1,2} = -2.0468 \pm 12.9858i$, $\lambda_3 = 4.3654$. (Another stable equilibrium point is $(5.7072, 12.7845, 1.5971)$.) Initial values are $(0, 0, -0.1)$. For system (29) all conditions of Theorem 8 are valid. There is a homoclinic orbit connected at $O = (0,0,0)$. This orbit and the entire chaotic attractor are disposed below plane XOY.

(5) Another system with a chaotic attractor as (see Fig. 7):

$$\begin{cases} \dot{x}(t) = 3x + 10y + 2x^2 + 2xy + xz \\ \qquad - y^2 + z^2, \\ \dot{y}(t) = -10x + 3y - x^2 + 2xy + yz \\ \qquad + y^2 - z^2, \\ \dot{z}(t) = z^2 + 2xz + 3yz. \end{cases} \tag{30}$$

It is obtained from system (15) with linear transformations of coordinates x and y. Here eigenvalues of the Jacobi matrix at the point $(0,0,0)$ are $\lambda_{1,2} = 3 \pm 10i$, $\lambda_3 = 0$; at the equilibrium

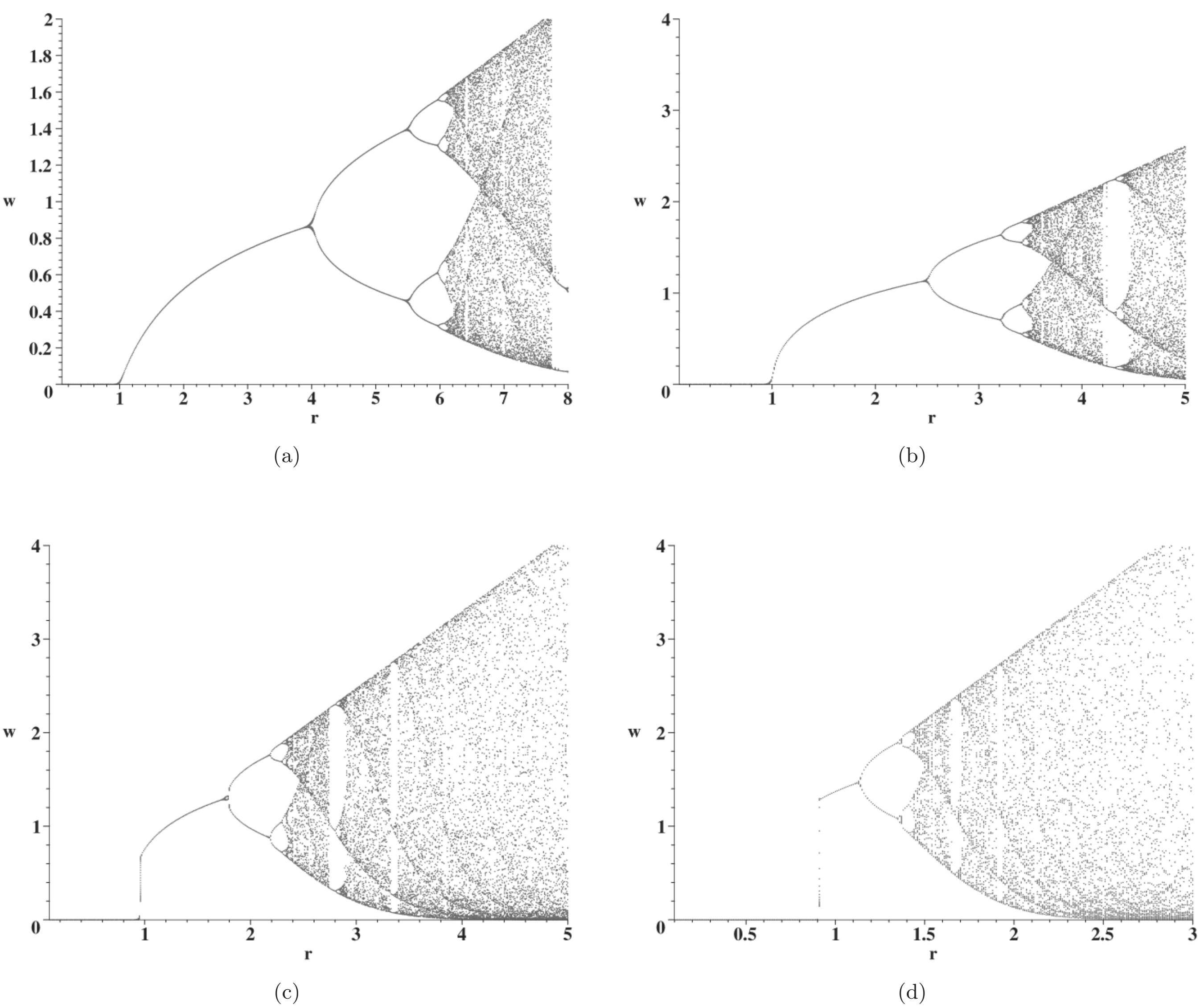

Fig. 8. The bifurcation diagram of map (24) at $\gamma = 1$ and different values of p. (a) $p = 0$, (b) $p = 1$, (c) $p = 1.5$ and (d) $p = 2$.

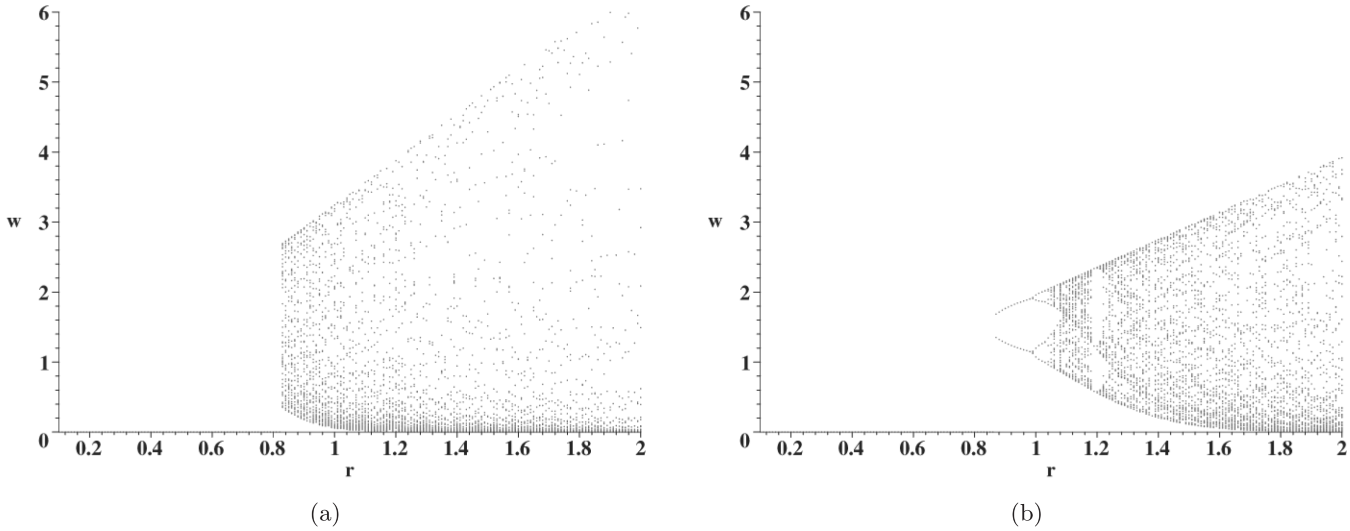

Fig. 9. The bifurcation diagram of map (24) at $p = 2$ and different values of γ. (a) $\gamma = 0$ and (b) $\gamma = 0.5$.

point $(-6.5443, 6.7266, 0)$ eigenvalues are $\lambda_{1,2} = -3.1797 \pm 15.1922i$, $\lambda_3 = 7.0912$, and at the equilibrium point $(-0.7152, -0.3071, 2.3517)$ eigenvalues are $\lambda_{1,2} = 4.7180 \pm 10.7237i$, $\lambda_3 = -1.8651$. For system (30) all conditions of Theorem 8 are valid. There is a homoclinic orbit connected at $O = (0, 0, 0)$. This orbit and the entire chaotic attractor are disposed below plane XOY.

(6) All systems (26)–(30) have either periodic or homoclinic orbits. As shown in Sec. 4, a behavior of these systems is determined by discrete map (24). Bifurcation diagrams of map (24) are below (see Figs. 8 and 9). These diagrams show a chaotic behavior of map (24) [and systems (26)–(30)] at some values of parameters $r > 0$, $\gamma > 0$, and p.

The analysis of Figs. 8 and 9 shows that with either growth of the parameter p or diminishing parameter γ, a scenario is seen of the chaotic behavior of system (15) deviating from the classic Feigenbaum scenario of the period doubling bifurcation.

6. Conclusion

The present work is a continuation of article [Belozyorov, 2011a]. As compared to the paper [Belozyorov, 2011a], the following new results are obtained.

(1) Theorems 6 and 7 guaranteeing the existence in system (15) of different types of invariant sets are included.
(2) Theorem 8 on the existence of homoclinic orbits in the case of several equilibriums (including singular) of system (15) is proved.
(3) For system (15) at the same parameters, the presence of a few homoclinic orbits is shown.
(4) The new exponential 1D discrete map (24) determining the chaotic behavior of system (15) is built.
(5) In Theorem 9, the chaotic behavior of map (24) for some values of parameters is proved.

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