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Sharp Constant in Jackson's Inequality with Modulus of Smoothness for Uniform Approximations of Periodic Functions

S. A. Pichugov*

Dnepropetrovsk National Technical University of Railroad Communications

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Abstract—It is proved that, in the space $C_{2\pi}$, for all $k, n \in \mathbb{N}$, $n > 1$, the following inequalities hold:

$$\left(1 - \frac{1}{2n}\right)\frac{k^2+1}{2} \le \sup_{\substack{f \in C_{2\pi}\\ f \ne \mathrm{const}}} \frac{e_{n-1}(f)}{\omega_2(f, \pi/(2nk))} \le \frac{k^2+1}{2}.$$

where $e_{n-1}(f)$ is the value of the best approximation of f by trigonometric polynomials and $\omega_2(f,h)$ is the modulus of smoothness of f. A similar result is also obtained for approximation by continuous polygonal lines with equidistant nodes.

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Suppose that

- $C_{2\pi}$ is the space of (2π)-periodic real-valued continuous functions f with norm

$$\|f\| = \max\{|f(x)| : x \in \mathbb{R}\};$$

- $e_{n-1}(f) = \inf_{T_{n-1}} \|f - T_{n-1}\|$ is the value of the best approximation of f in this space by trigonometric polynomials T_{n-1} of degree at most $n-1$, $n \in \mathbb{N}$;
- $\omega_2(f,h) = \sup_{|t| \le h} \|\Delta_t^2 f\|$ is the value of the modulus of smoothness of f at a point h, $h \ge 0$, where

$$\Delta_t^2 f(x) = f(x+t) + f(x-t) - 2f(x)$$

is the second difference of f at a point x with step t.

Theorem 1. *For all $k, n \in \mathbb{N}$, $n > 1$, the following inequalities hold*:

$$\left(1 - \frac{1}{2n}\right)\frac{k^2+1}{2} \le \sup_{\substack{f \in C_{2\pi}\\ f \ne \mathrm{const}}} \frac{e_{n-1}(f)}{\omega_2(f, \pi/(2nk))} \le \frac{k^2+1}{2}. \tag{1}$$

Corollary 1. *For all $k \in \mathbb{N}$, the following relations hold*:

$$\sup_n \sup_{\substack{f \in C_{2\pi}\\ f \ne \mathrm{const}}} \frac{e_{n-1}(f)}{\omega_2(f, \pi/(2nk))} = \frac{k^2+1}{2}.$$

*E-mail: s.a.pichugov@mail.ru

Upper bounds for the values of best approximations of functions in terms of the values of their moduli of continuity of various orders are called the *Jackson's inequalities*. Well-known results concerning sharp Jackson inequalities (i.e., inequalities with sharp constants) for functions of one variable can be found in [1]–[8]. In particular, in the case $k = 1$, inequalities (1) were proved in [9], [10] (upper bound) and [6] (lower bound). Also note the paper [11] in which, for other values of the argument of the modulus of smoothness, upper bounds for sharp constants were obtained.

Suppose that M is an arbitrary subspace in $\mathrm{C}_{2\pi}$ containing constants,

$$e(f; M) = \inf\{\|f - g\| : g \in M\}$$

is the value of the best approximation of f by the subspace M,

$$\mathcal{W}^2 = \{f \in \mathrm{C}_{2\pi} : f' \in AC,\ f'' \in \mathrm{C}_{2\pi},\ \|f''\| \le 1\},$$

and $e(\mathcal{W}^2; M)$ is the value of the best approximation of the class $\mathcal{W}^2$ by the subspace M.

Lemma 1. 1) *For any f from $\mathrm{C}_{2\pi}$, the following inequality holds*:

$$e(f; M) \le \frac{1}{2}\inf_{h>0}\biggl(1 + \frac{2e(\mathcal{W}^2; M)}{h^2}\biggr)\omega(f, h); \tag{2}$$

2) *for any $\delta > 0$, the following inequalities hold*:

$$\frac{\delta^2}{2} \le \sup_{\substack{f\in \mathrm{C}_{2\pi}\\ f\ne\mathrm{const}}} \frac{e(f; M)}{\omega_2(f, (2e(\mathcal{W}^2; M))^{1/2}/\delta)} \le \frac{\delta^2 + 1}{2}. \tag{3}$$

Proof of Lemma 1. For $h > 0$, suppose that

$$\mathcal{S}_h(f, x) = \frac{1}{h}\int_{-h/2}^{h/2} f(x + t)\, dt$$

is the Steklov mean of f with step h, and

$$\mathcal{S}_{h^2}(f, x) := S_h(\mathcal{S}_h f, x) = \frac{1}{h^2}\int_{-h}^{h}(h - |t|)f(x + t)\, dt$$

is the Steklov mean of f of second order. Then

$$|f(x) - \mathcal{S}_{h^2}(f, x)| \le \frac{1}{h^2}\int_0^h (h - t)|\Delta_t^2 f(x)|\, dt,$$

$$\|f - \mathcal{S}_{h^2} f\| \le \frac{1}{h^2}\int_0^h (h - t)\omega_2(f, t)\, dt \le \frac{1}{2}\omega_2(f, h).$$

Further,

$$\|D^2(\mathcal{S}_{h^2} f)\| = \biggl\|\frac{\Delta_h^2 f}{h^2}\biggr\| \le \frac{\omega_2(f, h)}{h^2}, \qquad e(\mathcal{S}_{h^2} f; M) \le \frac{1}{h^2}\omega_2(f, h)e(\mathcal{W}^2; M).$$

Now, to find an upper bound for the approximation value $e(f; M)$, we use the intermediate approximation of f by smoother functions $\mathcal{S}_{h^2} f$:

$$e(f; M) \le \|f - \mathcal{S}_{h^2} f\| + e(\mathcal{S}_{h^2} f; M) \le \frac{1}{2}\biggl(1 + \frac{2}{h^2}e(\mathcal{W}^2; M)\biggr)\omega_2(f, h). \tag{4}$$

Since the value of h is arbitrary, we obtain (2). Note that this method was used in [9] to find estimate (4) for the approximation by polynomials.

If we put $h = (2e(\mathcal{W}^2; M))^{1/2}/\delta$ in (4), then we obtain the upper bound in (3). We obtain the lower bound in (3) by restricting ourselves to the approximation of smooth functions f from $\mathrm{C}_{2\pi}$ and using the inequality $\omega_2(f, h) \le \|f''\|h^2$.

Lemma 1 is proved. □

Proof of Theorem 1. In the case of approximation by trigonometric polynomials using the Akhiezer–Krein–Favard theorem (see, for example, [7]), we obtain

$$\sup_{f\in\mathcal{W}^2} e_{n-1}(f) = \frac{\pi^2}{8n^2}, \tag{5}$$

and then the upper bound in (3) is of the form

$$\sup_{\substack{f\in \mathrm{C}_{2\pi}\\ f\neq\mathrm{const}}} \frac{e_{n-1}(f)}{\omega_2(f,\pi/(2n\delta))} \le \frac{\delta^2+1}{2}. \tag{6}$$

Let us show that, for $\delta\in\mathbb{N}$, this estimate cannot be improved for all n.

To find lower bounds for the Jackson constants in the construction of the following functions we use an idea of Korneichuk [1], [2], which was realized in [6] for the moduli of smoothness for $\delta=1$.

Let us fix

$$k,n\in\mathbb{N},\quad n>1,\qquad \varepsilon\in\left(0,\frac{1}{2}\right],$$

and set

$$x_0=0,\qquad x_\nu=\nu h-(n-\nu)\beta,\quad \nu=1,\dots,n,\qquad h=\frac{\pi}{n},\qquad \beta\in\left(0,\frac{4\varepsilon}{n^2(k^2+1)}\right).$$

By construction,

$$x_{\nu+1}-x_\nu=h+\beta,\qquad x_n=\pi.$$

Consider an arbitrary function f from $\mathrm{C}_{2\pi}$ satisfying the conditions

$$f(-x)=f(x),\qquad f(0)=0,\qquad f(x_\nu)=(-1)^{\nu+1}\frac{k^2+1}{2},\quad \nu=1,\dots,n. \tag{7}$$

To find a lower bound for $e_{n-1}(f)$, we use the polynomial

$$T_{n-1}(x)=\frac{k^2+1}{2n}\,\frac{\sin(n-1/2)x}{2\sin(x/2)}.$$

For $\nu=0,1,\dots,n$, we have (see [1], [2])

$$f(x_\nu)-T_{n-1}(x_\nu)=(-1)^{\nu+1}\left(\frac{k^2+1}{2}-\frac{k^2+1}{4n}\right)+\mu_\nu,$$

where $|\mu_\nu|<\varepsilon$; hence, taking into account the fact that f is even and using the Vallée-Poussin theorem, we obtain

$$e_{n-1}(f)\ge\frac{k^2+1}{2}\left(1-\frac{1}{2n}\right)-\varepsilon. \tag{8}$$

Let us now define the function $f(x)$ on the whole axis so that, along with conditions (7), the following condition also holds:

$$\omega_2\left(f,\frac{\pi}{2nk}\right)=1. \tag{9}$$

First, let us construct $f(x)$ on the closed interval $[x_1,\gamma]$, where $\gamma=(3/2)(h+\beta)-n\beta$ is the midpoint of the closed interval $[x_1,x_2]$, specifying it the polygonal line uniquely defined by its values at the nodes:

$$\begin{gathered} f(\gamma)=0,\qquad f\left(x_1+j\frac{h}{2k}\right)=\frac{k^2+1}{2}-\frac{j^2}{2},\quad j=0,\dots,k,\\ f\left(x_1+j\frac{h}{2k}+\frac{\beta}{2}\right)=\frac{k^2+1}{2}-\frac{(j+1)^2}{2},\quad j=0,\dots,k-1. \end{gathered} \tag{10}$$

Let us continue $f(x)$ to the closed interval $[\gamma, x_2]$ as an odd function with respect to the point γ:

$$f(\gamma+x) = -f(\gamma-x), \qquad x \in \left[0, \frac{x_2-x_1}{2}\right]. \tag{11}$$

Further, we set

$$\begin{gathered} f(x) = -f(x-h-\beta), \qquad x \in [x_2, \pi], \\ f(x) = \max\{0; f(2x_1 - x)\}, \qquad x \in [0, x_1], \\ f(-x) = f(x), \qquad x \in [-\pi; 0], \\ f(x+2\pi) = f(x). \end{gathered} \tag{12}$$

This defines the continuous 2π-periodic function satisfying conditions (7). It is easy to see that condition (9) also holds: since $f(x)$ is a polygonal line, it follows that, to calculate its modulus of smoothness, it suffices to calculate the increments of the function f at its nodes.

Since ε is arbitrary, relations (8) and (9) imply the lower bound of the Jackson constant in (1).

Theorem 1 is proved. □

Remark 1. In the proof of Lemmas 1, we did not use the specific properties of the metric of $C_{2\pi}$; in particular, relations (3) are also valid in the space $L_1[0, 2\pi]$. Further, the analog (5) of the Akhiezer–Krein–Favard Theorem also holds in $L_1[0, 2\pi]$ (see, for example, [7]). Therefore, in the space $L_1[0, 2\pi]$, the following upper bound similar to (6) is also valid:

$$\sup_{\substack{f \in L_1[0,2\pi] \\ f \neq \mathrm{const}}} \frac{e_{n-1}(f)_{L_1}}{\omega_2(f, \pi/(2n\delta))_{L_1}} \le \frac{\delta^2+1}{2}.$$

However, we do not know the exact values of the Jackson constants for the moduli of smoothness in this space for any $\delta > 0$.

Remark 2. Suppose that $X_{n-1,2}(f)$ are the Favard sums of degree $n-1$ of order 2 for the function f (see, for example, [7]). Then

$$\widetilde{\mathscr{L}}_{n-1}(f) := \mathcal{S}_{(\pi/(2nk))^2} \circ X_{n-1,2}(f)$$

is the best linear method for approximating functions among all linear polynomial methods $\mathscr{L}_{n-1}$ in the sense that, for any $k \in \mathbb{N}$,

$$\sup_n \inf_{\mathscr{L}_{n-1}} \sup_{\substack{f \in C_{2\pi} \\ f \neq \mathrm{const}}} \frac{\|f - \mathscr{L}_{n-1}(f)\|}{\omega_2(f, \pi/(2nk))} = \sup_n \sup_{\substack{f \in C_{2\pi} \\ f \neq \mathrm{const}}} \frac{\|f - \widetilde{\mathscr{L}}_{n-1}(f)\|}{\omega_2(f, \pi/(2nk))} = \frac{k^2+1}{2}.$$

This immediately follows from the proof of the upper bound in Theorem 1 and the fact (see, for example, [7]) that

$$e_{n-1}(\mathcal{W}^2) = \sup_{f \in \mathcal{W}^2} \|f - X_{n-1,2}(f)\|.$$

It is easy to calculate the multipliers of the method $\widetilde{\mathscr{L}}_{n-1}$: if f_ν are the complex Fourier coefficients of f and

$$\widetilde{\mathscr{L}}_{n-1}(f, x) = \sum_{|\nu|<n} \alpha_k\left(\frac{\nu}{n}\right) f_\nu e^{i\nu x},$$

then

$$\alpha_k(t) = 4k^2 \left(\sin\frac{\pi}{4k} t\right)^2 \frac{\cos(\pi t/2)}{(\sin(\pi t/2))^2}, \qquad |t| \le 1.$$

In particular (see [9], [10], [6]),

$$\alpha_1(t) = 1 - \left(\tan\frac{\pi}{4}t\right)^2, \qquad \alpha_2(t) = \frac{1-(\tan(\pi t/4))^2}{(\cos(\pi t/8))^2}.$$

Let us also consider the approximation of functions by the subspace $\mathcal{S}_{2n}$ of periodic continuous polygonal lines with the $2n$ equidistant nodes

$$y_\nu = \frac{\pi}{2n} + \frac{\nu\pi}{n}, \qquad \nu \in \mathbb{Z},$$

on the period $[-\pi, \pi]$.

Theorem 2. *For all $k, n \in \mathbb{N}$, $n > 1$, the following inequalities hold*:

$$\left(1 - \frac{1}{2n}\right)\frac{k^2+1}{2} \le \sup_{\substack{f\in C_{2\pi}\\ f\neq \mathrm{const}}} \frac{e(f;\mathcal{S}_{2n})}{\omega_2(f,\pi/(2nk))} \le \frac{k^2+1}{2}. \tag{13}$$

Corollary 2. *For all $k \in \mathbb{N}$, the following relations hold*:

$$\sup_n \sup_{\substack{f\in C_{2\pi}\\ f\neq \mathrm{const}}} \frac{e(f;\mathcal{S}_{2n})}{\omega_2(f,\pi/(2nk))} = \frac{k^2+1}{2}.$$

Proof. Since (see [11])

$$e(\mathcal{W}^2;\mathcal{S}_{2n}) = \frac{\pi^2}{8n^2},$$

we see that the upper bound in (13) follows from (3).

To find the lower bound, we consider the approximation of the function f constructed in the proof of Theorem 1 (see (7), (10)–(12)). We shall use the duality relation for approximation by splines of minimal deficiency [12]; in our particular case, this relation can be expressed as

$$e(f;\mathcal{S}_{2n}) = \sup\left\{\int_{-\pi}^{\pi} f(x)\,dg_1(x) : \operatorname{Var} g_1(x) \le 1,\ g_2(y_\nu) = \mathrm{const},\ \nu \in \mathbb{Z}\right\}, \tag{14}$$

where $g_2(x)$ is the antiderivative of $g_1(x)$, which is zero in the mean, and $\operatorname{Var} g_1(x)$ is the variation of $g_1(x)$ on the period.

To find the lower bound for $e(f;\mathcal{S}_{2n})$, we construct a piecewise constant function $g_1(x)$ as follows: first, we define the auxiliary function $\psi(x)$ on the period $[-\pi,\pi]$ as an even continuous polygonal line with zeros at the points y_ν and the vertices at the points x_ν.

For $x \in [0,\pi]$, let

$$\psi(x) := c_\nu(x - y_\nu), \quad x \in [x_{\nu-1}, x_\nu], \qquad \nu = 1,\dots,n.$$

The continuity condition at the point $x_{\nu+1}$ means that

$$c_{\nu+1} = -c_\nu \frac{\pi/(2n) - (n-(\nu+1))\beta}{\pi/(2n) + (n-(\nu+1))\beta}.$$

Set $c_1 = -1$; then, for $\nu = 2,\dots,n$,

$$c_\nu = (-1)^\nu \prod_{j=1}^{\nu-1} \frac{\pi/(2n) - (n-j)\beta}{\pi/(2n) + (n-j)\beta}. \tag{15}$$

The function $\psi'(x)$ is piecewise constant and

$$\operatorname{Var}\psi'(x) = 4\sum_{\nu=1}^{n} |c_\nu|.$$

Set

$$g_2(x) = \frac{\psi(x) - \psi_0}{4\sum_{\nu=1}^{n} |c_\nu|}, \qquad \text{where} \quad \psi_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(x)\, dx.$$

Then $g_2(x)$ is zero in the mean, $g_2(y_\nu) = \text{const}$, $\nu \in \mathbb{Z}$, and $\operatorname{Var} g_1(x) = 1$. Since

$$|c_\nu - c_{\nu+1}| = |c_\nu| + |c_{\nu+1}|,$$

(see (15)), it follows from (14) that

$$e(f; \mathcal{S}_{2n}) \ge \int_{-\pi}^{\pi} f(x)\, dg_1(x) = \frac{1}{4\sum_{\nu=1}^{n} |c_\nu|} 2\bigg(\sum_{\nu=1}^{n-1} |c_\nu - c_{\nu+1}| + |c_n|\bigg)\frac{k^2+1}{2}$$

$$= \frac{1}{4\sum_{\nu=1}^{n} |c_\nu|}\bigg(4\sum_{\nu=1}^{n} |c_\nu| - 2|c_1|\bigg)\frac{k^2+1}{2} = \bigg(1 - \frac{1}{2\sum_{\nu=1}^{n} |c_\nu|}\bigg)\frac{k^2+1}{2}.$$

Equality (15) implies that $|c_\nu| \to 1$ as $\beta \to 0$. This yields the lower bound in (13). Theorem 2 is proved. □

It is possible that the assertion of Theorem 2 remains valid in the case of approximation by splines of minimal deficiency and any order $r \in \mathbb{N}$. In this case, the upper bound in (13) holds and it suffices only to prove the lower bound.

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